Mathematics to finance & back again: three bond market case studies

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Three practical problems

- Portfolio risk estimation
  - From value at risk to spectral risk measures
- Default risk modeling
  - From Merton models to strategic interaction
- Term structure models
  - From relative value graphs to weak foliations

- Quantitative analytics project life cycle
  1. Implement the classical solution
  2. Encounter practical difficulties
  3. Google for more recent research—see references
  4. Encounter theoretical difficulties

Portfolio risk estimation

Value-at-risk vs. actual losses

Company	Daily VaR	Holding period, confidence level	Reported losses 7/1/98-9/30/98
Lehman	$14m	1 day, 95%	$60m
Citicorp	$25m	1 day, 97.7%	$200m
Chase	$28m	1 day, 99%	$200m
J P Morgan	$29m	1 day, 95%	na
Dean Witter	$39m	1 day, 99%	$110m
Salomon	$44m	1 day, 95%	$300m
Goldman Sachs	$47m	1 day, 95%	na
Merrill Lynch	$60m	1 week, 99%	$135m
Bankers' Trust	$83m	10 days, 99%	$350m
Deutsche Bank	$87m	10 days, 99%	na
CSFB	$140m	10 days, 99%	$55m
UBS	$193m	10 days, 99%	$600m

Definition of risk measure

- $(\Omega, \mathcal{F}, P)$ probability space ('outcomes')
- V a set of $\mathcal{F}$-measurable rv's ('positions')
- A risk measure is a map $\rho: V \to \mathbb{R}$

- Example: value-at-risk
  - Fix a confidence level α (e.g. 99%)
  - Time horizon determines choice of P, or maybe $(\Omega, \mathcal{F}, P)$
  - VaR is the quantile $\rho(X) = -\sup \{x \in \mathbb{R}: P[X \geq x] \geq \alpha\}$
- Example: expected shortfall
  - Avg loss in worst $(1 - \alpha)$% of n iid copies of X, as $n \to \infty$

VaR, expected shortfall and tail risk

- Portfolio 1: $1bn
  - Single firm with 0.5% dflt prob, 0% loss given dflt
- Portfolio 2: $1bn
  - 1000 such firms, with uncorrelated default processes

- 99% VaR
  - $0m for portfolio 1, $5m for portfolio 2
- 99% expected shortfall
  - $500m for portfolio 1, $5m for portfolio 2

What is a good risk measure?

1. Monotonicity: $X \square Y \square\ \square(X) \geq \square(Y)$
2. Positive homogeneity: $\square(aX) = a\square(X)$
3. Translation invariance: $\square(X + a) = \square(X) - a$
4. Subadditivity: $\square(X + Y) \square \square(X) + \square(Y)$
   - Diversification never hurts; disaggregation doesn't help
5. Law invariance: $P[X \square t] = P[Y \square t]\ \square\ t \square R \square\ \square(X) = \square(Y)$
   - Risk of *X* depends only on its statistical properties
6. Comonotonic additivity: $\square(fX + gX) = \square(fX) + \square(gX)$
   - *f*, *g* nondecreasing and *fX*, *gX* $\square$ *V*
   - *f*, *g* represent two kinds of exposure to the same risk
     - e.g. different contingent claim payoffs

VaR vs. expected shortfall

- Industry standard risk measures
  - VaR: 1, 2, 3, 5, 6; but not 4
  - Scenario analysis: 1, 2, 3, 4, 6; but not 5

- Coherent risk measure: 1, 2, 3, 4
  - Representation theorem
    - A risk measure ρ is coherent iff it can be calculated as the maximum loss over a (possibly infinite) set of probabilities $Q \ll P$ (scenarios)
- Spectral risk measure: 1, 2, 3, 4, 5, 6
  - Characterization theorem
    - Expected shortfall is the smallest spectral majorant of VaR
  - Representation theorem
    - Spectral risk measures are weighted combinations of expected shortfalls, where confidence levels are weighted via a ‘spectrum’ which can be interpreted as a subjective risk aversion function

Informative risk measurement

- Questions that portfolio managers ask
  - How much could I lose?
  - What should I worry about the most?
  - Are there any hidden correlations?
- Statistical estimation vs. identifying scenarios of interest
  - Is a law invariant risk measure inherently uninformative?

- A first step: marginal risk contributions
  - Think of sub-portfolios/risk factors/strategies
    - What is the risk contribution of a piece of the portfolio?
    - $X = \sum w_i X_i$ and want to estimate $w_i(\partial\rho/\partial w_i)$
  - For spectral risk measures, this is a well-behaved concept
    - Especially in the case of expected shortfall

Default risk modeling

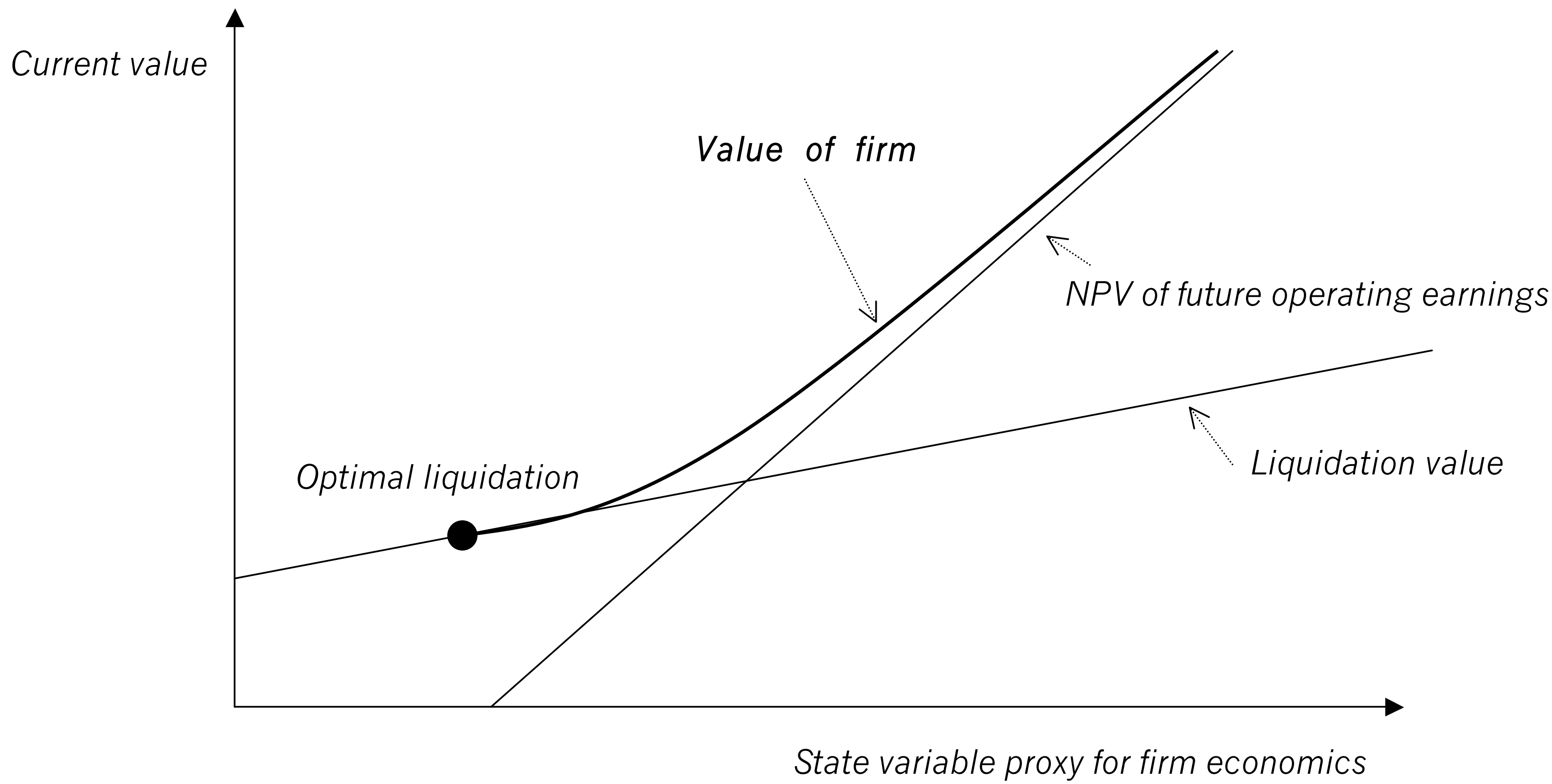
Price history, before & after default

Merton model vs. game theory

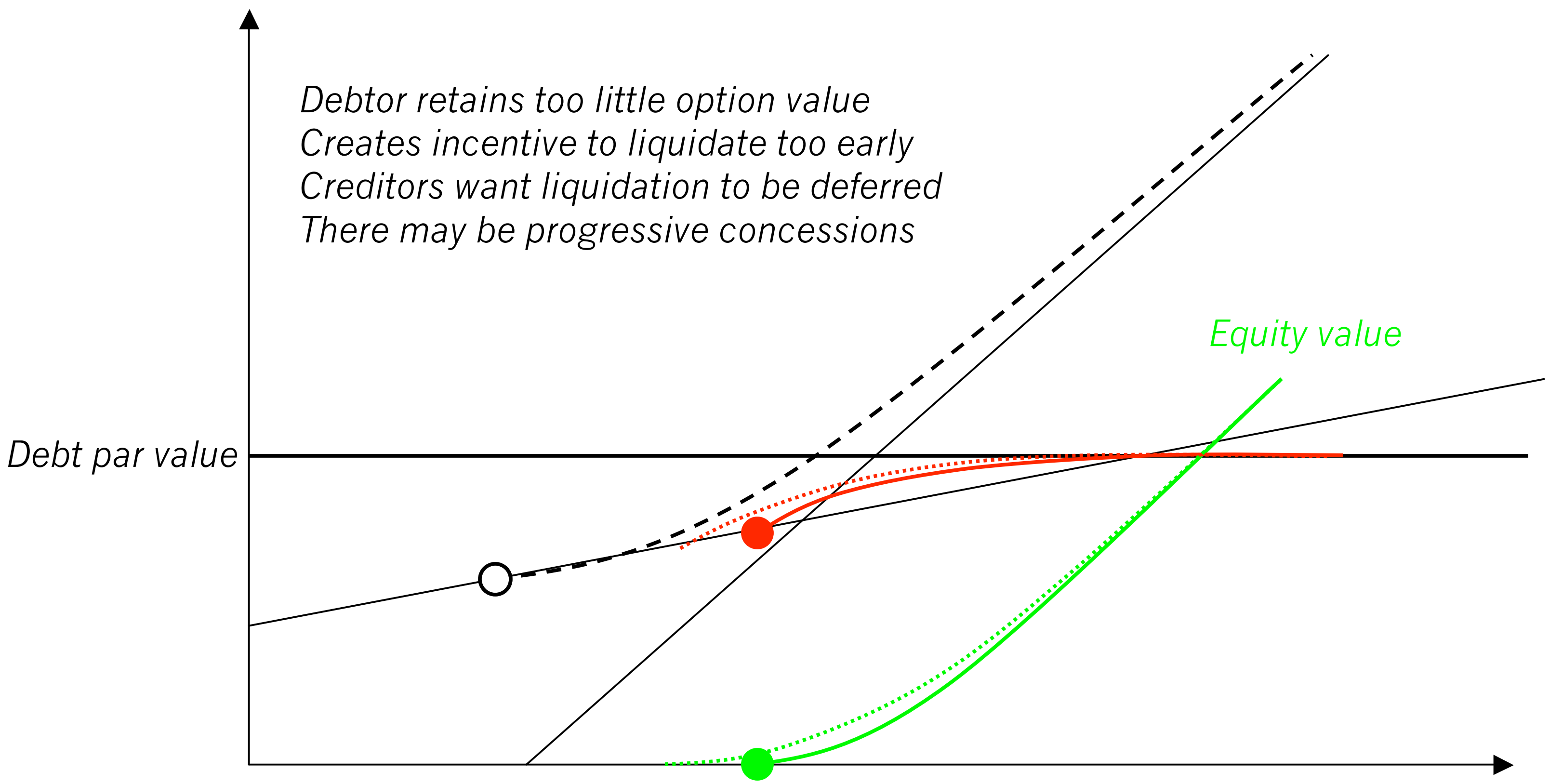
- Merton model
  - Model the firm value $V = D + E$ as a stochastic process
  - Default occurs when V crosses a 'default threshold'
    - Often an *ad hoc* definition; e.g. KMV: ST debt + 0.5 x LT debt

- Game-theoretic critique
  - Credit risk ≠ risk of an atomic 'default' event
  - Almost always a multi-stage process
    - Distress → Default → Chapter 11 filing → Restructuring
    - Distress → Default → Chapter 7 filing → Liquidation
    - Various combinations of the above, over a period of years (TWA)
  - Options exist, strategic interaction occurs at every stage

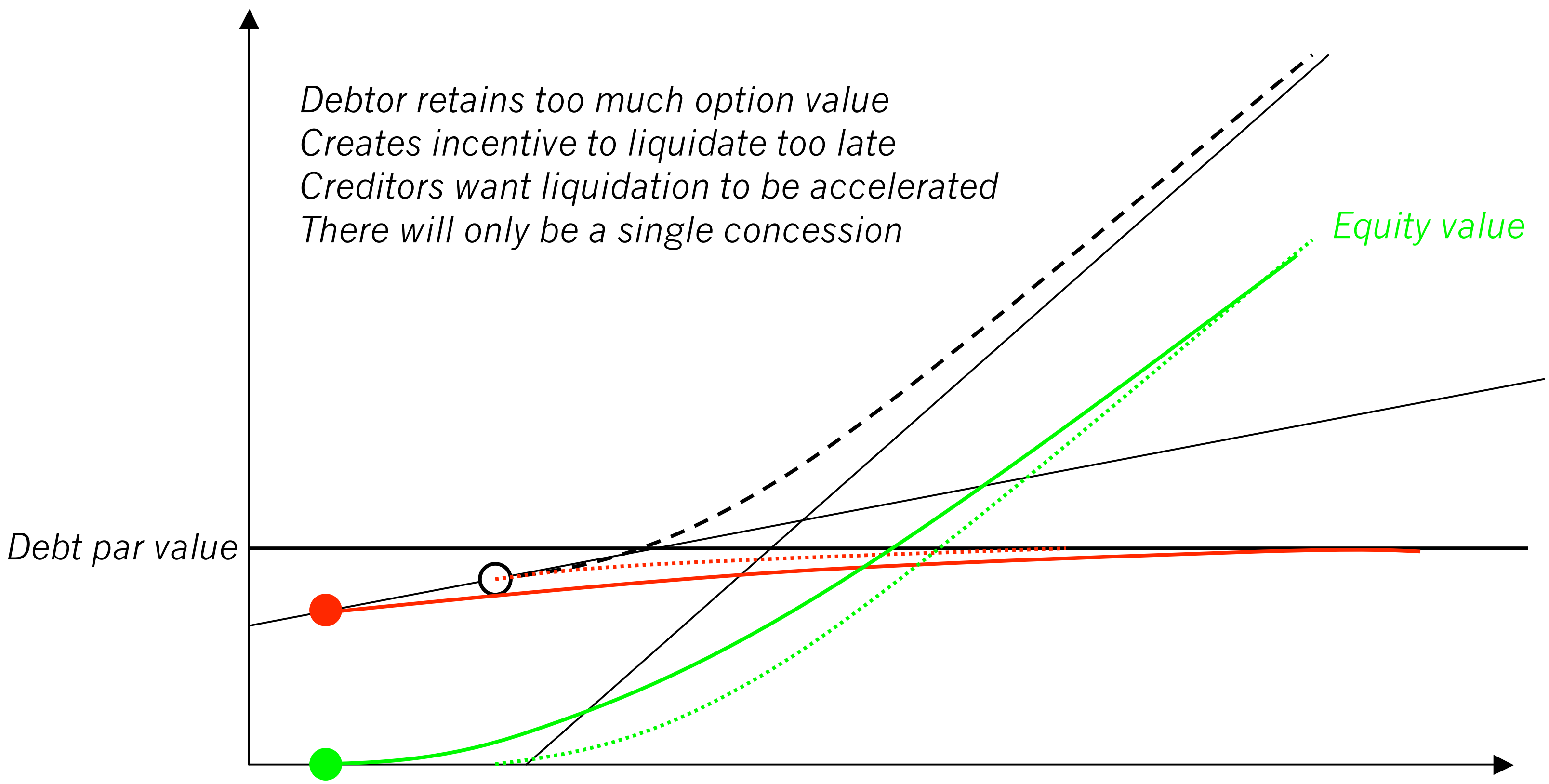
Optimal liquidation, no debt



Bankruptcy point, high leverage



Bankruptcy point, low leverage

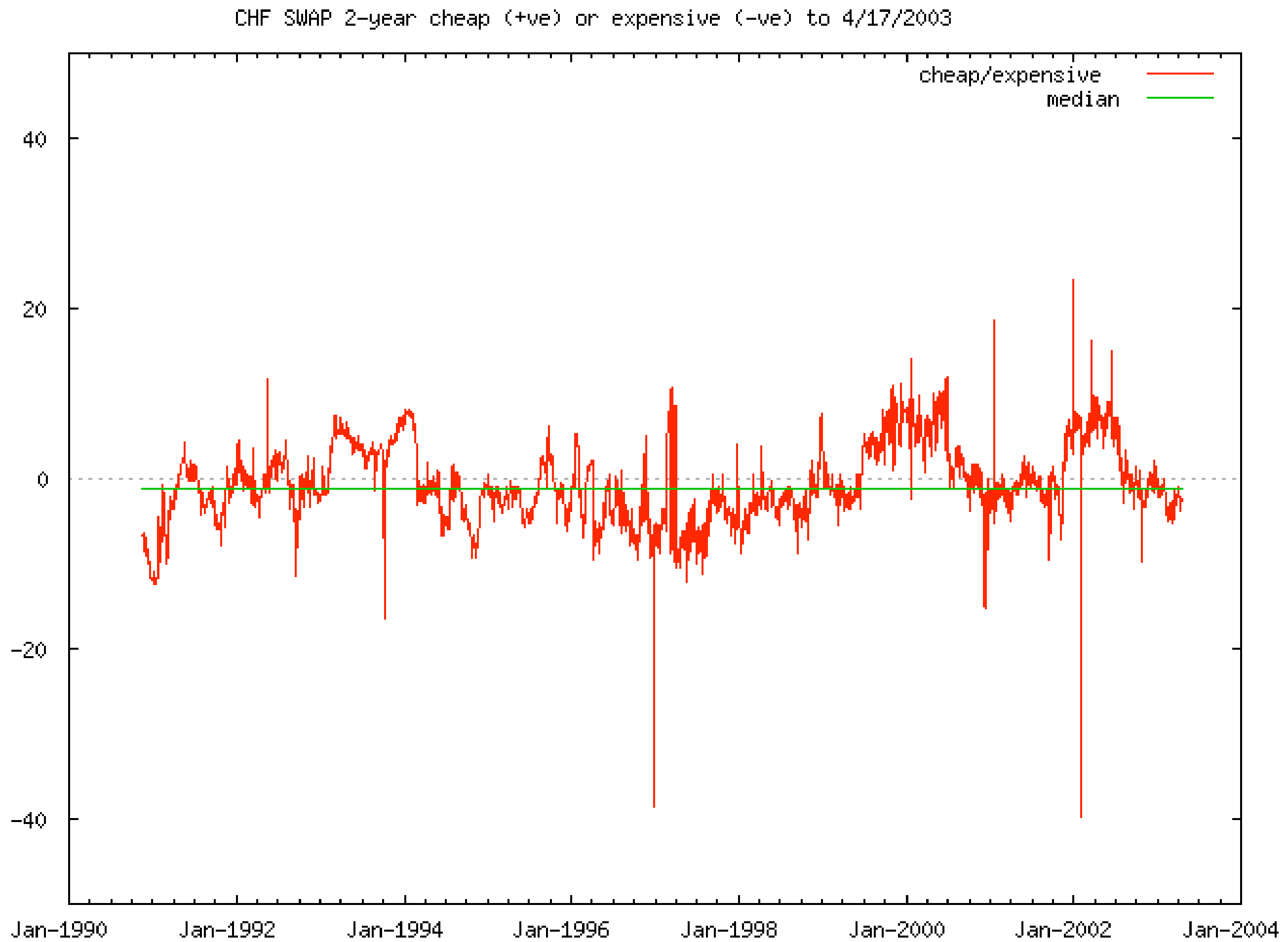


Some further problems

- Analytic tractability
  - Differential games are hard to solve
- Non-zero bankruptcy costs
  - Furthermore, costs depend on strategy adopted
- Non-stationary capital structure
  - Short and long term debt mix, subordination,...
- Different classes of creditors
  - Bondholders, banks, trade creditors,...
- Industry level considerations
  - Risk of multiple bankruptcies lowers liquidation values

Term structure models

Yield curve relative value analysis



Consistency questions

- Most models assume a family of ‘good’ curves
  - E.g. Nelson-Siegel forward curves $f(x) = a + be^{-dx} + cxe^{-dx}$
  - Find best fit to observed yield curve
  - Interpret discrepancies as deviations from fair value
- When is this procedure consistent?
  - Consistency relative to a given term structure model
    - Start with a forward curve in the family
      - Look at its stochastic evolution as specified by the model
      - Do you stay inside the family?
    - If consistent, can structure relative value trades to hedge out risk
  - Example: no nontrivial model is consistent with NS!
- Conversely: model family of ‘good’ curves?
  - E.g. closed form solutions of standard models

HJM framework, FDR problem

- Evolution of $f_t = f_t(x)$, forward curve at time t
  - $\mathrm{d}f_t = \square\,\mathrm{d}t + \square \cdot \mathrm{d}W_t$, where
    - Drift term $\square = Af_t + \square$ where $A = \mathrm{d}/\mathrm{d}x$ generates time shift
    - Vector $\square = (\square^i)$ of state-dependent volatility functions $\square^i(x) = \square^i(t, f_t)(x)$
    - And $\square(x) = \square(t, f_t)(x)$ is the notorious HJM drift term $\square^i(x) \cdot (\int^x \square^i(y)\mathrm{d}y)'$
  - Solutions of this SDE live in an ∞-dim Hilbert space H
    - SDE is driven by a d-dim Wiener process $W = (W^i)$
    - Yet it might conceivably sweep out an ∞-dim subset of H

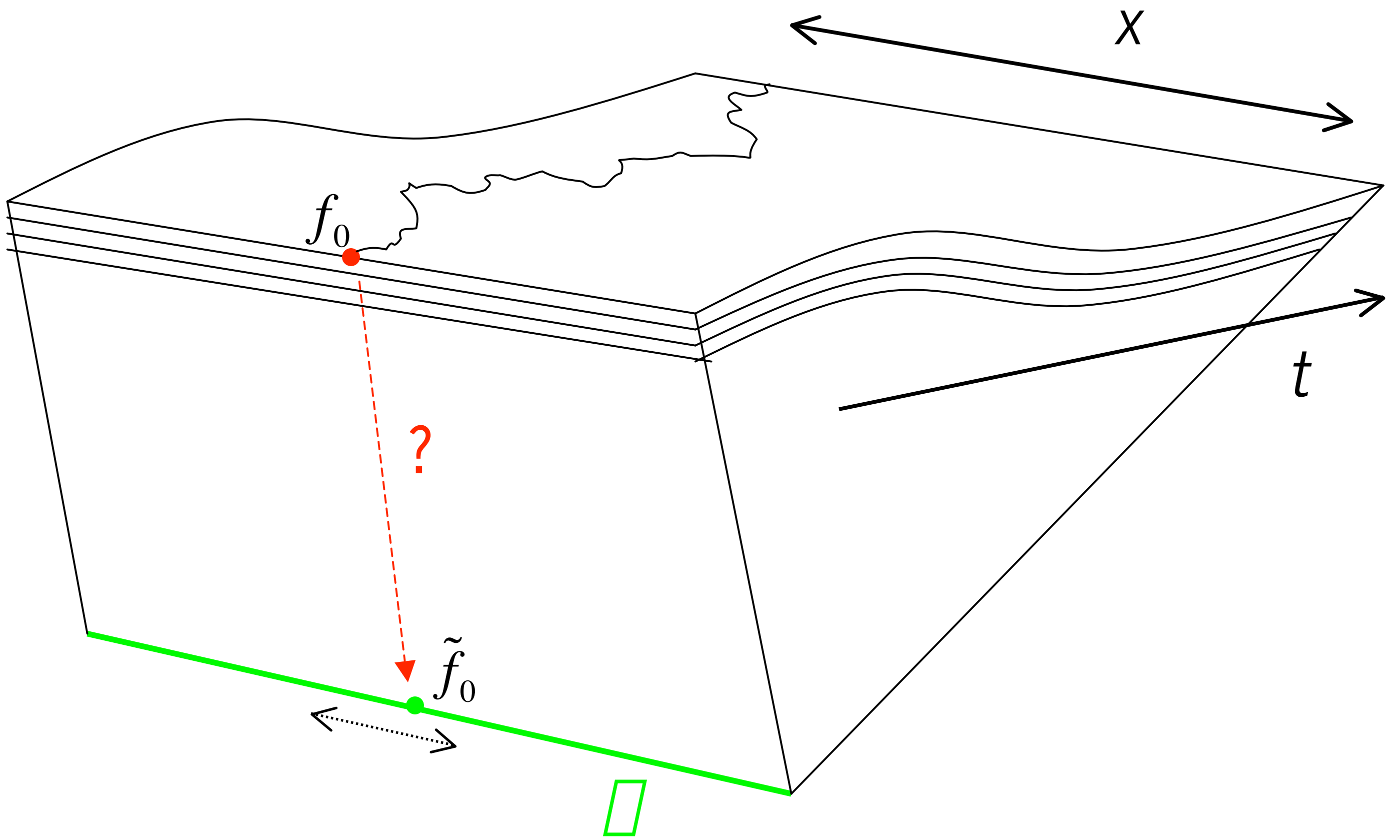
- When is there a finite dimensional realization?
  - i.e. solutions confined to a finite dimensional manifold $\square$ H
  - In fact, have finite dim state space $Y \square R_{\geq 0} \times R^n$
    - First variable is time evolution, other variables represent state of economy
    - Want to find a stochastic process on Y, and a function $Y \square H$ mapping y_t to f_t
  - 'Generically', i.e. extends to a nbd of a given initial fwd curve f_0

Local and global results

- Apply ‘Frobenius theorem’ on the Fréchet space $D(A^\infty)$ □ H
  – If □ and the □^{i} generate a k-dim Lie algebra…
    - This must be true for all initial forward curves in a nbd of f_0
  – …then flows integrate to get k-dim tangential manifolds
    - Complication: in □ direction, there may only be a semiflow
  – This defines a ‘weak foliation’ (analog of finite-dim case)
    - This is true off a singular set □ on which □ lies in the span of the □^{i}
    - Hopefully, □ is a singular leaf of dim k–1 (time evolution is redundant on □)
    - These are the fwd curves for which the model is time-homogeneous
  – Regularity thm: solutions to SDE must in fact live inside $D(A^\infty)$

- When are the conditions satisfied on all of $D(A^\infty)\backslash$□ ?
  – Answer: iff the term structure model is affine!

The geometrical picture



Some outstanding problems

- What is the fitting process?
  - NB: not fitting model parameters, but state variables
  - What is the meaning of OLS estimation? Other methods?
- What are suitable models?
  - PCA suggests you need either 2 or 3 factors
  - Long rate constrained by odd theorems of Dybvig et al.
- What is the appropriate setting?
  - Fréchet spaces are too hard
  - Why do we want a more abstract setting?
    - Define leaf spaces of 'singular weak foliations' in ∞-dim setting
    - E.g. stochastic volatility as semi-direct product of Lie algebroids?

Selected references

- Portfolio risk estimation
  - Dirk Tasche, "Expected shortfall and beyond", *Journal of Banking and Finance* **26(7)** 1519-1533.
- Default risk modeling
  - Pierre Mella-Barral, "The dynamics of default and debt reorganization", *Review of Financial Studies* **12(3)** 535-578.
- Term structure models
  - Damir Filipovic & Josef Teichmann, "On the geometry of the term structure of interest rates", *Proceedings A of the Royal Society*, to appear (2003).