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A Uniform Approach to Domain Theory in Realizability Models

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We propose a uniform way of isolating a subcategory of predomains within the category of modest sets determined by a partial combinatory algebra (PCA). Given a *divergence* on a PCA (which determines a notion of partiality), we identify a candidate category of predomains, the *well-complete objects*. We show that, whenever a single *strong completeness axiom* holds, the category satisfies appropriate closure properties. We consider a range of examples of PCAs with associated divergences and show that in each case the axiom does hold. These examples encompass models allowing a "parallel" style of computation (e.g. by interleaving), as well as models that seemingly allow only "sequential" computation, such as those based on term-models for the lambda-calculus. Thus our approach provides a uniform approach to domain theory across a wide class of realizability models. We compare our treatment with previous approaches to domain theory in realizability models. It appears that no other approach applies across such a wide range of models.

1. Introduction

The familiar category of modest sets, which provides models of many interesting type theories, is constructed from the partial combinatory algebra (PCA), $\mathbb{K}$, of the natural numbers under Kleene application. As is well known, the construction is completely general and one can define an analogous category for any PCA. This generality is not spurious as there are many other interesting PCAs whose properties are quite different from those of $\mathbb{K}$, for example PCAs based on untyped λ-terms.

One can think of an arbitrary PCA, A, as a primitive untyped programming language (in a very abstract sense as, for example, A may be uncountable). The category of modest sets provides a type-theoretic framework built over the untyped language. Indeed modest sets provide a very general notion of datatype, and the morphisms between two modest sets correspond to the (total) computable functions between the associated datatypes (where, of course, the concept of computability is determined by A). The rich categorical structure of the category of modest sets then offers everything that is needed for interpreting a wide range of powerful type theories — see, e.g., (Longo and Moggi 1991).

All this works well for those type theories that type only total functions. But real programming languages allow arbitrary recursively defined functions and recursively defined datatypes, and consequently include programs that do not terminate. Traditionally such features are modelled using the techniques of domain theory. In this paper we provide a uniform approach to modelling them in categories of modest sets. To do this, we identify appropriate structure for doing "domain theory" in such "realizability models".

In Sections 2 and 3 we introduce PCAs and define the associated "realizability" categories of *assemblies* and *modest sets*. Next, in Section 4, we prepare for our development of domain theory with an analysis of nontermination. Previous approaches have used (relatively complicated) categorical formulations of partial maps for this purpose. Instead, motivated by the idea that A provides a primitive programming language, we consider a simple notion of "diverging" computation within A itself. This leads to a theory of *divergences* from which a notion of (computable) partial function is derived together with a lift monad classifying partial functions.

The next task is to isolate a subcategory of modest sets with sufficient structure for supporting analogues of the usual domain-theoretic constructions. First, we expect to be able to interpret the standard constructions of total type theory in this category, so it should inherit cartesian-closure, coproducts and the natural numbers from modest sets. Second, it should interact well with the notion of partiality, so it should be closed under application of the lift functor. Third, it should allow the recursive definition of partial functions. This is achieved by obtaining a *fixpoint object* in the category, as defined in (Crole and Pitts 1992). Finally, although there is in principle no definitive list of requirements on such a category, one would like it to support more complicated constructions such as those required to interpret polymorphic and recursive types.

The central part of the paper (Sections 5, 6, 7 and 9) is devoted to establishing such a category of "predomains". As in classical domain theory, our treatment is based on a notion of chain completeness. However, there are several differences (for one, we do not work in an order-theoretic setting). Indeed a considerable amount of work is required to obtain a category satisfying the (first three) conditions discussed above. Nevertheless we do obtain such a category, the category of *well-complete* objects, and we show that it does indeed have the desired properties, whenever a single *strong completeness axiom* is satisfied.

The generality of our approach is demonstrated by considering, in Section 8, a number of diverse PCAs (together with divergences upon them) and showing that the strong completeness axiom holds in all cases. These examples encompass models allowing a "parallel" style of computation (e.g. by interleaving), as well as models that seemingly allow only "sequential" computation, such as those based on term-models for the lambda-calculus. In Section 10, we compare our work with other approaches previously proposed for isolating categories of predomains in realizability models. It appears that none of these approaches achieves such a general level of applicability.

In contrast to much previous work, we adopt a very concrete "external" style of presentation in this paper. Mainly we prove results by "programming" in a PCA. In this respect our approach is close to that of (Freyd *et al.* 1990), but we work in a much more general setting. We hope that the external approach will make our work accessible to computer scientists who are not well versed in the "internal logic" style of synthetic domain theory, as adopted in, e.g., (Phoa 1990; Hyland 1990; Taylor 1991). Indeed for the major part of the paper we do not consider the realizability topos over A at all. However, although such an approach is perfectly feasible for developing the basic results of this paper, it does have its limitations. For more complicated constructions, such as the interpretation of polymorphic and recursive types, it appears that our external approach would become prohibitively overburdened with detail. In Section 11, we conclude the paper by sketching how the internal logic of the realizability topos may be usefully exploited to perform such constructions.

2. Partial combinatory algebras

In this section we define the notion of *partial combinatory algebra (PCA)* on which the realizability categories will be based. This notion appears in many places, e.g., (Barendregt 1984; Beeson 1985).

Definition 2.1. (Partial applicative structure) A *partial applicative structure*, $(A, \cdot)$, consists of a set A equipped with a *partial* operation, $\cdot$, from $A \times A$ to A.

We shall often refer to such a structure simply as A, leaving the application operation implicit. We say that A is *total* if the application operation is total.

Let A be any partial applicative structure. We use $a, b, c, \dots$ to range over elements of A, and $x, y, z, \dots$ to range over a countably infinite set of variables. We use $e, e', \dots$ to range over the set $\mathcal{E}(A)$ of *formal expressions* over A, which is generated by the grammar:

$$e \quad ::= \quad a \mid x \mid (ee').$$

We shall often omit the parenthesis around juxtaposition, assuming it to associate to the left. The substitution, $e[e'/x]$, of e' for x in e is defined in the obvious way.

We say an expression is closed if it contains no variables. There is an evident partial "evaluation" function from closed expressions to A defined by interpreting juxtaposition as the partial application operation. If e is closed we write $e\downarrow$ to mean that evaluation is defined on e, and $e\uparrow$ to mean that it is not. Note that if $ee'\downarrow$ then both $e\downarrow$ and $e'\downarrow$. If e, e' are closed, we write $e = e'$ to mean that e and e' are both defined and denote the same value (strict equality), and $e \simeq e'$ to mean if either e or e' is defined then so is the other and they denote the same value (Kleene equality). More generally, if e, e' are expressions whose variables are among $x_1, \dots, x_n$, we write

$$\begin{array}{lll} e\downarrow & \text{to mean} & e[\vec{a}/\vec{x}]\downarrow \ \text{ for all } a_1, \dots, a_n \in A; \\ e \simeq e' & \text{to mean} & e[\vec{a}/\vec{x}] \simeq e'[\vec{a}/\vec{x}] \ \text{ for all } a_1, \dots, a_n \in A. \end{array}$$

We use this notation to give the main definition of the section.

Definition 2.2. (Partial combinatory algebra) A *partial combinatory algebra (PCA)* is a partial applicative structure $(A, \cdot)$ such that there exist distinct elements $\mathbf{k}, \mathbf{s} \in A$ satisfying:

$$\mathbf{k}xy \simeq x, \qquad \mathbf{s}xyz \simeq xy(xz), \qquad \mathbf{s}xy\downarrow.$$

Note that, as $x\downarrow$, the Kleene equality of the first equation can be replaced by strict equality. Clearly a total PCA is just an ordinary combinatory algebra (Hindley and Seldin 1986). We now present some examples of both partial and total PCAs.

(i) (The Kleene PCA) Consider the set $\mathbb{N}$ of natural numbers equipped with *Kleene application*: $m \cdot n \simeq \{m\}(n)$, where $\{m\}$ denotes the partial recursive function coded by m under some standard enumeration. The existence of $\mathbf{k}, \mathbf{s}$ with the required properties is an immediate consequence of the S-m-n theorem, see e.g. (Cutland 1980, Chapter 4). We refer to this PCA as $\mathbb{K}$, for Kleene. Note that $\mathbb{K}$ is *not* total.

(ii) (Closed λ-term PCAs) Let Λ^0 denote the set of closed terms of the untyped lambda-calculus. Let $\mathcal{T}$ be any λ-theory (Barendregt 1984). The quotient $\Lambda^0/\mathcal{T}$ forms an evident total PCA with application defined by $[M] \cdot [N] = [M(N)]$.

(iii) (Dcpo-based λ-models) Let A be any nontrivial dcpo (i.e. a poset with at least two elements, including a least one, and with sups of all directed subsets) for which it holds that $[A \to A] \lhd A$ (i.e. A has its space of all continuous endofunctions as a retract via continuous functions). Let $\psi : A \to [A \to A]$ be a selected retraction from A to $[A \to A]$. Then A forms a total PCA with application defined by $ab = (\psi(a))(b)$.

The three (families of) PCAs presented above will remain our principal examples throughout the paper. However, there are many other interesting PCAs, for example: recursion-theoretic PCAs based on Turing degrees; closed λ-term PCAs which are partial (e.g. ones based on operational termination in the call-by-value λ-calculus (Plotkin 1975)); λ-term models based on open terms; genuinely partial dcpo-based combinatory algebras defined using retracts of spaces of *partial* continuous endofunctions; sub-PCAs of dcpo-based PCAs obtained by restricting to the *effective* elements; and models of the untyped lambda calculus based on stable functions. Clearly this list is far from exhaustive. Some of the above examples are considered in more detail in (Longley 1995).

We consider a PCA as providing an abstract primitive programming language. Indeed many of the results in this paper will be proved by "programming" in a PCA. To establish notation, we review some basic definability properties of PCAs. For fuller details see (Beeson 1985; Longley 1995).

Let A be a PCA. Let $\mathbf{k}$ and $\mathbf{s}$ be chosen elements of A with the properties required by Definition 2.2 (note that $\mathbf{k}$ and $\mathbf{s}$ are not necessarily uniquely determined by these properties). We write $\mathbf{i}$ for the element $\mathbf{skk}$. Clearly, for all $a \in A$, it holds that $\mathbf{i}a = a$.

Proposition 2.3. (Combinatory completeness) For any $e \in \mathcal{E}(A)$ there is a formal expression $\lambda^* x.\, e \in \mathcal{E}(A)$, whose variables are just those of e excluding x, such that $(\lambda^* x.\, e){\downarrow}$ and $(\lambda^* x.\, e)x \simeq e$.

Proof We define $\lambda^* x.\, e$ by induction on the structure of e: $\lambda^* x.\, x$ is $\mathbf{i}$; $\lambda^* x.\, y$ is $\mathbf{k}y$ if y is a constant or variable other than x; and $\lambda^* x.\, uv$ is $\mathbf{s}(\lambda^* x.\, u)(\lambda^* x.\, v)$. □

If $e \in \mathcal{E}(A)$, the notation $\lambda^* x.\, e$ is to be understood as meta-notation for some expression e' involving $\mathbf{k}$ and $\mathbf{s}$. Note that we always have $(\lambda^* x.\, e){\downarrow}$ whatever e. We shall make use of obvious abbreviations, such as writing $\lambda^* xy.\, e$ for $\lambda^* x. \lambda^* y.\, e$ (i.e. the formal expression obtained by first translating $\lambda^* y.\, e$ to e', then translating $\lambda^* x.\, e'$ to e'').

The λ^* notation provides a useful λ-calculus for defining elements of PCAs. However, one must take care when using this notation to reason about equality between elements, as β-conversion is only applicable in restricted cases. This issue is discussed thoroughly in (Longley 1995), where a formal λ-calculus of meta-expressions is presented and valid instances of β-conversion are identified. In the present paper, for lack of space, we shall simply claim that various equalities hold between expressions defined using the λ^* notation. All these claims can be justified using the restricted β-rules of (Longley 1995), but they are also quite straightforward to prove from first principles.

Proposition 2.4. (Pairing and booleans)
(i) There exist elements *pair*, *fst*, *snd* $\in A$ such that

$$\mathit{pair}\ x\ y\!\downarrow, \quad \mathit{fst}\,(\mathit{pair}\ x\ y) \ = \ x, \quad \mathit{snd}\,(\mathit{pair}\ x\ y) \ = \ y$$

(ii) There exist elements *true*, *false*, *if* $\in A$ such that

$$\mathit{if}\ x\ y\!\downarrow, \quad \mathit{if}\ \mathit{true}\ y\ z \ = \ y, \quad \mathit{if}\ \mathit{false}\ y\ z \ = \ z$$

PROOF (i) Take $\mathit{pair} = \lambda^* xyz.zxy$, $\mathit{fst} = \lambda^* v.v(\lambda^* xy.x)$, $\mathit{snd} = \lambda^* v.v(\lambda^* xy.y)$.
(ii) Take $\mathit{true} = \lambda^* yz.y$, $\mathit{false} = \lambda^* yz.z$, $\mathit{if} = \lambda^* xyz.xyz$. □

We shall normally write $\langle M, M' \rangle$ as an abbreviation for $\mathit{pair}\ M\ M'$.

Definition 2.5. (Curry numerals) For each $n \in \mathbb{N}$, we define an element $\overline{n} \in A$ by $\overline{0} = \mathbf{i}$, and $\overline{n+1} = \langle \mathit{false}, \overline{n} \rangle$.

Proposition 2.6. There exist elements *succ*, *pred*, *iszero* $\in A$ such that, for all $n \in \mathbb{N}$,

$$\mathit{succ}\,\overline{n} \ = \ \overline{n+1}, \quad \mathit{pred}\,\overline{0} \ = \ \overline{0}, \quad \mathit{pred}\,\overline{n+1} \ = \ \overline{n},$$
$$\mathit{iszero}\,\overline{0} \ = \ \mathit{true}, \quad \mathit{iszero}\,\overline{n+1} \ = \ \mathit{false}$$

PROOF Take $\mathit{succ} = \lambda^* x.\langle \mathit{false}, x \rangle$, $\mathit{iszero} = \mathit{fst}$, $\mathit{pred} = \lambda^* x.\mathit{if}\,(\mathit{iszero}\ x)\,\overline{0}\,(\mathit{snd}\ x)$. □

Notice that $\overline{n_1} = \overline{n_2}$ implies $n_1 = n_2$ and so any PCA contains a faithful copy of the natural numbers. Other representations of the natural numbers are also possible (for example, in the case $A = \mathbb{K}$ we could simply take $\overline{n} = n$). However, the above representation is a convenient one that works in an arbitrary PCA, and any other reasonable representation will be equivalent to it.

Proposition 2.7. (Fixed-point combinator) There exists $\mathbf{z} \in A$ such that

$$\mathbf{z}\,y\!\downarrow, \quad (\mathbf{z}\,y)\,z \ \simeq \ y\,(\mathbf{z}\ y)\,z$$

PROOF Define $\mathbf{z} = (\lambda^* xyz.\, y(xxy)z)(\lambda^* xyz.\, y(xxy)z)$. □

Proposition 2.8. (Primitive recursion) There exists an element *rec* $\in A$ such that

$$\mathit{rec}\ x\ f\ \overline{0} \ = \ x, \qquad \mathit{rec}\ x\ f\ \overline{n+1} \ \simeq \ f\ \overline{n}\ (\mathit{rec}\ x\ f\ \overline{n}).$$

PROOF Let $R = \lambda^* r\,x\,f\,m.\ \mathit{if}\,(\mathit{iszero}\ m)\,(\mathbf{k}x)\,(\lambda^* y.\ f\,(\mathit{pred}\ m)\,(r\ x\ f\,(\mathit{pred}\ m)\,\mathbf{i}))$ and take $\mathit{rec} = \lambda^* x\,f\,m.\,(\mathbf{z}\,R)xfm\mathbf{i}$. □

Using the constant *rec*, it is easy to see how any primitive recursive function can be represented in a PCA. In fact, with a little more work using $\mathbf{z}$, one can show how to represent any *partial recursive* function, see (Beeson 1985, Section VI.2.8). Thus the programming language of the PCA is Turing complete.

3. Assemblies and modest sets

In this section we use a PCA, A, to construct two categories: the category, **Ass**, of *assemblies* and the category, **Mod**, of *modest sets*. Very roughly, the category of assemblies provides a setting for considering computable functions between sets, where the notion of computability is provided by the PCA A. The category of modest sets is the full-subcategory of **Ass** consisting of "datatype-like" sets. Computationally, **Mod** will be our category of primary interest. Indeed we shall not use **Ass** in any substantial way until Section 11. However, **Ass** does provide a natural ambient category with sufficient categorical structure for performing the constructions used in the paper. We should mention that **Ass** is itself a full subcategory of another important category, the *realizability topos*, **RT**, derived from A. We shall not define **RT** as its construction is rather intricate, see (Hyland 1982; Hyland *et al.* 1980). Nevertheless, we shall discuss relationships between **Ass**, **Mod** and **RT** when appropriate.

Definition 3.1. (Assembly) (i) An *assembly* X on A consists of a set $|X|$ together with a function assigning to each element $x \in |X|$ a non-empty subset $\|x\|$ of A. If $a \in \|x\|$, we say that *a realizes x*. To avoid ambiguity we sometimes write $\|x\|_X$ for $\|x\|$.
(ii) Suppose X, Y are two assemblies on A. A function $f : |X| \to |Y|$ is said to be *tracked* by an element $r \in A$ if for all $x \in |X|$ and $a \in A$ we have

$$a \in \|x\|_X \quad \Longrightarrow \quad ra\downarrow \text{ and } ra \in \|f(x)\|_Y.$$

A *morphism* $f : X \to Y$ is a function $f : |X| \to |Y|$ that is tracked by some $r \in A$.

It is easy to see that assemblies on A and their morphisms form a category: for any assembly X the identity function $\mathrm{id}_{|X|}$ is tracked by $\mathbf{i}$; and if $f : X \to Y$ and $g : Y \to Z$ are tracked by r and t respectively, then $g \circ f$ is tracked by $\lambda^* a.\, t(ra)$. We write $\mathbf{Ass}(A)$ for this category, or **Ass** when A is understood from the context. The category $\mathbf{Ass}(\mathbb{K})$ is frequently called the category of *ω-sets* in the literature, see e.g. (Longo and Moggi 1991).

Intuitively one may think of an assembly X as a set, $|X|$, each of whose elements, x, comes equipped with an associated collection, $\|x\|$ of "computational representations". Then a morphism from X to Y is a function that is "computable" in the sense that there exists a "program" (i.e. an element of A) whose action on representations respects (i.e. tracks) the function.

We now display some of the categorical structure of **Ass**. Our notation and terminology is fairly standard. By *cartesian category* we mean a category with finite products, and by *bicartesian category* we mean one with finite products and coproducts. A (bi)cartesian category is *(bi)cartesian closed* if, for every object X, the endofunctor $X \times (-)$ has a right-adjoint. See (Mac Lane 1971) or (Barr and Wells 1995) for full definitions. Recall also the notion of a *natural numbers object (nno)*, as in, e.g., (Barr and Wells 1995).

Theorem 3.2. The category **Ass** is bicartesian-closed with finite limits and nno.

Proof We describe the structure explicitly, leaving the routine verification of the details to the reader. For a more thorough account of this material (worked out for the PCA $\mathbb{K}$) we refer the reader to (Hyland 1988) or (Phoa 1992b).

The terminal object $\mathbf{1}$ is given by $|\mathbf{1}| = \{*\}$, $\|*\| = A$. Binary products are given by

$$|X \times Y| \;=\; |X| \times |Y| \qquad \|(x,y)\|_{X\times Y} \;=\; \{\, \langle a, b\rangle \mid\; a \in \|x\|_X,\; b \in \|y\|_Y\},$$

and the two projections are tracked by *fst* and *snd*.

The initial object $\mathbf{0}$ is defined by $|\mathbf{0}| = \emptyset$. The sum of X and Y is given by

$$|X + Y| \;=\; |X| + |Y|$$
$$\|(0, x)\|_{X+Y} \;=\; \{\, \langle \mathit{true}, a\rangle \mid\; a \in \|x\|_X\} \qquad \|(1, y)\|_{X+Y} \;=\; \{\, \langle \mathit{false}, a\rangle \mid\; a \in \|y\|_Y\},$$

and the two injections are tracked by $\lambda^* x.\, \langle \mathit{true}, x\rangle$ and $\lambda^* y.\, \langle \mathit{false}, y\rangle$

The exponential Y^X is given by

$$|Y^X| \;=\; \mathbf{Ass}(X, Y) \qquad\qquad \|f\|_{Y^X} \;=\; \{r \in A \mid r \text{ tracks } f\},$$

and the evaluation function $X \times Y^X \to Y$ is tracked by $\lambda^* a.\, (\mathit{snd}\ a)\, (\mathit{fst}\ a)$.

Given $f, g : X \rightrightarrows Y$, the equalizer $e : Z \to X$ is defined by

$$|Z| \;=\; \{x \in |X| \mid fx = gx\} \qquad \|x\|_Z \;=\; \|x\|_X,$$

where e is the inclusion function, tracked by $\mathbf{i}$.

For the nno, let N be given by

$$|N| \;=\; \mathbb{N}, \qquad \|n\|_N \;=\; \{\overline{n}\}$$

and let $z : \mathbf{1} \to N$ and $s : N \to N$ be the morphisms tracked by $\lambda^* a.\, \overline{0}$ and *succ* respectively. If X is an assembly and we have maps $\mathbf{1} \to X \to X$ tracked by r, t respectively, then the unique mediating morphism $N \to X$ is tracked by $\mathit{rec}\ (r\, \mathbf{i})\, (\mathbf{k}\, t)$. □

In fact much more holds. For example, **Ass** has coequalizers, is regular and is locally cartesian closed; but we shall not use any of this additional structure. However, we do note that **Ass** is *well pointed*, i.e. the functor $\mathbf{Ass}(\mathbf{1}, -) : \mathbf{Ass} \to \mathbf{Set}$ is faithful. Also a morphism is mono if and only if it is injective; and similarly it is epi if and only if it is surjective.

One object of **Ass** that will be important later is the object of booleans, $\mathbf{1} + \mathbf{1}$. For convenience we shall use the isomorphic object, $\mathbf{2}$, defined by:

$$|\mathbf{2}| \;=\; \{\mathit{tt}, \mathit{ff}\} \qquad\qquad \|\mathit{tt}\| \;=\; \{\mathit{true}\} \qquad\qquad \|\mathit{ff}\| \;=\; \{\mathit{false}\}.$$

The following definition will sometimes be used to specify objects of **Ass**.

Definition 3.3. (Regular subobject) The *regular subobject* of an assembly X *determined by* a subset $Z \subseteq |X|$ is the assembly Y defined by $|Y| = Z$ and $\|y\|_Y = \|y\|_X$.

The terminology is motivated by the construction of equalizers in the proof of Theorem 3.2. Indeed one can show that any mono is isomorphic (in the standard sense) to the inclusion from a regular subobject (as defined above) if and only if it is a regular mono.

The category **Ass** with its structure identified above will provide the ambient category for all the constructions in this paper. For the interested reader, we remark that it arises naturally as the full subcategory of double-negation-separated objects in **RT** (the realizability topos based on A). The structure identified above is all inherited from **RT** with the sole exception of regular subobjects. The regular subobjects of **Ass** do not correspond to the regular subobjects of **RT**, as in a topos every subobject is regular. Rather, regular subobjects in **Ass** correspond to double-negation-closed subobjects (of double-negation-separated objects) in **RT**.

However, rather than looking at **Ass** from the perspective of a larger category, we shall work in the opposite direction and isolate a well behaved subcategory of **Ass** to act as a category of datatypes. From a computational point of view, it is natural to consider the full subcategory of those assemblies in which no two distinct elements share a common realizer. This corresponds to a reasonable notion of datatype, because an element of a datatype should be determined by any of its computational representations. Moreover the morphisms between such datatypes are really "computed" by the elements tracking them, in the sense that each tracking element determines its function. (This property is not shared by **Ass** which, because of the possibility of sharing realizers, has **Set** as a full subcategory.) All this motivates the following definition.

Definition 3.4. (Modest set) An assembly X is called a *modest set* if, for all $x, x' \in |X|$, it holds that $x \neq x'$ implies $\|x\| \cap \|x'\| = \emptyset$.

We write $\mathbf{Mod}(A)$ (again normally **Mod**) for the full subcategory of $\mathbf{Ass}(A)$ consisting of modest sets.

We now show that **Mod** has rich categorical structure, all inherited from **Ass**. First some additional terminology. Let $\mathcal{C}'$ be a full subcategory of a category $\mathcal{C}$ closed under isomorphisms (i.e. such that X in $\mathcal{C}'$ and $X \cong Y$ in $\mathcal{C}$ implies Y in $\mathcal{C}'$). We say that $\mathcal{C}'$ is an *exponential ideal* of $\mathcal{C}$ if, for any objects X of $\mathcal{C}'$ and Y of $\mathcal{C}$, the object X^Y lies in $\mathcal{C}'$. We say that $\mathcal{C}'$ is *closed under subobjects* if whenever there is a mono $X \to Y$ in $\mathcal{C}$ and Y lies in $\mathcal{C}'$ then so does X.

Theorem 3.5. The category **Mod** is bicartesian closed with finite limits and nno and the inclusion in **Ass** preserves this structure. Further, **Mod** is an exponential ideal of **Ass** and is closed under subobjects.

PROOF It is easily checked that **Mod** is closed under subobjects (hence under isomorphisms) and that the constructions in the proof of Theorem 3.2 produce modest sets under the relevant conditions. □

Next, we consider an important fact about **Mod**, in which it differs from **Ass**. Consider the full subcategory, **ModP**, of **Mod** whose objects satisfy, for all $x \in |X|$, $x = \|x\|_X$ (this is just a way of defining the category of partial equivalence relations on A). The category **ModP** is easily seen to be both small and equivalent to **Mod**. Thus **Mod** is equivalent to a small category. This fact has dramatic consequences when viewed from the perspective of the associated realizability topos, **RT**. The remarkable fact about **ModP** is that it can be construed as an internal category in **RT** which, as well as being small, is "complete" in a suitable internal sense. This important property will be discussed further in Section 11. The reader is referred to (Hyland 1988; Hyland *et al.* 1990) for details.

The discussion in this section has been restricted to those properties of $\mathbf{Ass}(A)$ and $\mathbf{Mod}(A)$ that hold independently of the choice of A. However, when one looks more closely at the categories one sees that there many differences between them do emerge when one varies A. For example, if A is either the Kleene PCA, $\mathbb{K}$, or one of the closed λ-term PCAs discussed earlier then the morphisms from N to N in $\mathbf{Mod}(A)$ are exactly the total recursive functions, whereas if A is a dcpo based λ-model then every function on the natural numbers gives a morphism from N to N. In Section 8 we shall see that interesting differences between the first two examples arise when one considers issues of sequentiality. A general theory for relating different realizability models by means of exact functors is presented in (Longley 1995).

4. Divergences

In this section we consider a notion of partial function in which partiality is designed to correspond to some notion of diverging computation, typically nontermination. As our concept of computation is provided by a PCA, A, we look for a way of specifying diverging computations within A itself. In the case that A is non-total, then of course undefined application in A already gives a perfectly good notion of non-terminating computation. However, the existence of interesting total combinatory algebras prompts us to seek a more general notion. This leads us to specify conditions on a subset D of A under which it may reasonably be understood as corresponding to a subset of "diverging" computations. The following definition is due to the first author and was developed in his thesis (Longley 1995). The idea was inspired by a suggestion of Gordon Plotkin. A similar idea appears in (Abramsky and Ong 1993, §3) in the context of models for the lazy λ-calculus.

Definition 4.1. (Divergence) A *divergence* on a PCA A is a proper subset $D \subset A$ such that
(i) if A is total then $D \neq \emptyset$;
(ii) if $a \in D$ and $ab{\downarrow}$ then $ab \in D$.

The requirement that D be a proper subset is to ensure that not every computation diverges. Dually, the first condition guarantees that some diverging computations are possible. The second condition says that if a represents a diverging computation then ab cannot represent a converging computation. It follows that $\mathbf{i}$ is precluded from being in D (otherwise we would have $D = A$), as are $\mathbf{k}$ and $\mathbf{s}$.

Here are some simple examples of divergences.

(i) In any PCA, A, the singleton set $\{\mathbf{zk}\}$ is easily seen to be a divergence, because we have $\mathbf{zk}x \simeq \mathbf{k}(\mathbf{zk})x = \mathbf{zk}$.

(ii) If A is non-total, clearly $\emptyset$ is a divergence on A—this corresponds to the "natural" notion of diverging computation in the PCA.

(iii) Recall the notions of *unsolvable* λ-term (Barendregt 1984, §8.3) and *semi-sensible λ-theory* (Barendregt 1984, p. 77). Let $\mathcal{T}$ be any semi-sensible λ-theory. A natural divergence in $\Lambda^0/\mathcal{T}$ is given by the set $D = \{[M] \in \Lambda^0/\mathcal{T} \mid M \text{ unsolvable}\}$. This is indeed a divergence because $[\mathbf{\Omega}] \in D$ (where $\mathbf{\Omega} = (\lambda x.\, xx)\,(\lambda x.\, xx)$) and M unsolvable implies $M(N)$ unsolvable, see (Barendregt 1984, Corollary 8.3.4).

(iv) Let A be any nontrivial dcpo such that $[A \to A] \lhd A$. Then the singleton containing the least element, $\{\bot\}$, is a divergence in A (because the retraction from A to $[A \to A]$ preserves the least element).

Suppose given an arbitrary divergence $D \subset A$. We use this to specify a natural notion of partial function between assemblies. The idea is to use elements of D as tokens signalling undefinedness, and $\mathbf{i}$ as a token signalling definedness.

Definition 4.2. (Partial morphism) Given assemblies X, Y, we say that a partial function $f : |X| \rightharpoonup |Y|$ is *tracked* by $r \in A$ if:
(i) if $f(x){\uparrow}$ then, for all $a \in \|x\|$, $ra{\downarrow}$ implies $\mathit{fst}(ra) \in D$;
(ii) if $f(x){\downarrow}$ then, for all $a \in \|x\|$, $\mathit{fst}(ra) = \mathbf{i}$ and $\mathit{snd}(ra) \in \|f(x)\|$.
A *partial morphism* $f : X \rightharpoonup Y$ is a partial function $f : |X| \rightharpoonup |Y|$ that is tracked by some $r \in A$.

Any morphism $f : X \to Y$ tracked by r gives a partial morphism $f : X \rightharpoonup Y$ tracked by $\lambda^* a.\, \langle \mathbf{i},\, ra \rangle$. Also partial morphisms do compose: if $f : X \rightharpoonup Y$ and $g : Y \rightharpoonup Z$ are tracked by r and t respectively, then $\lambda^* a.\, \langle \mathit{fst}\,(r\,a)\ (\mathit{fst}\,(t\,(\mathit{snd}\,(r\,a)))),\ \mathit{snd}\,(t\,(\mathit{snd}\,(r\,a))) \rangle$ tracks $g \circ f$ (note how condition (ii) on divergences is used in verifying this). We write **pAss** for the category of assemblies and partial morphisms.

We believe that the above description of a partial category derived from a divergence has intuitive computational appeal. One might reasonably expect to interpret non-terminating programs within it. We now show that the same category arises more categorically as the Kleisli categories of a "lift"-monad. For basic information about monads consult (Mac Lane 1971).

First we define the lift functor $L : \mathbf{Ass} \to \mathbf{Ass}$. Its action on an object X is:

$$|LX| \;=\; \{\ulcorner x \urcorner \mid x \in |X|\} \cup \{\bot\}$$

$$\|\ulcorner x \urcorner\| \;=\; \{a \mid \mathit{fst}(a\mathbf{i}) = \mathbf{i},\ \mathit{snd}(a\mathbf{i}) \in \|x\|_X\} \qquad \|\bot\| \;=\; \{a \mid a\mathbf{i}{\downarrow} \Rightarrow \mathit{fst}(a\mathbf{i}) \in D\}.$$

Given a morphism $f : X \to Y$, define $Lf : LX \to LY$ by

$$Lf(\ulcorner x \urcorner) = \ulcorner f(x) \urcorner \qquad Lf(\bot) = \bot$$

which, if f is tracked by r, is tracked by $\lambda^* xy.\, \langle \mathit{fst}(x\mathbf{i}), r(\mathit{snd}(x\mathbf{i})) \rangle$. In fact note that the function $f \mapsto Lf$ in $|Y^X| \to |LY^{LX}|$ is tracked by $\lambda^* rxy.\, \langle \mathit{fst}(x\mathbf{i}), r(\mathit{snd}(x\mathbf{i})) \rangle$. We also observe that the function $\eta_X(x) = \ulcorner x \urcorner$ is tracked by $\lambda^* xy.\, \langle \mathbf{i},\, x \rangle$ to give a morphism $\eta_X : X \to LX$. Similarly there is a morphism $\mu_X : LLX \to LX$ given by:

$$\mu(\ulcorner\ulcorner x \urcorner\urcorner) = \ulcorner x \urcorner \qquad \mu(\ulcorner \bot \urcorner) = \mu(\bot) = \bot$$

which is tracked by $\lambda^* xy. \langle \mathit{fst}(x\mathbf{i})(\mathit{fst}(\mathit{snd}(x\mathbf{i})\mathbf{i})),\ \mathit{snd}(\mathit{snd}(x\mathbf{i})\mathbf{i}) \rangle$. The following fact is now easily verified:

Theorem 4.3. (L, η, μ) is a monad on **Ass**.

In fact, it is easy to check that (L, η, μ) is a *commutative strong* monad in the sense of (Kock 1970). The *strength* is a family of morphisms $X \times LY \to L(X \times Y)$ obtained from $(f \mapsto Lf) : Y^X \to LY^{LX}$ as in (Kock 1972). (Note that, because **Ass** is well-pointed the strength is determined by the monad.) Commutativity is an easily verified property of the strength. However, we will not explicitly use these facts in this paper.

Observe that **Mod** is closed under application of the lift functor L. So (L, η, μ) cuts down to a commutative strong monad on **Mod** too. Similarly, all the results presented below restrict to **Mod** in the obvious way.

Proposition 4.4. The Kleisli category of (L, η, μ) on **Ass** is isomorphic to **pAss**.

Proof One easily checks that $\mathbf{pAss}(X, Y) \cong \mathbf{Ass}(X, LY)$ and that the composition of partial morphisms corresponds to composition in the Kleisli category. □

Despite the computational relevance of the Kleisli category of the lift monad, in this paper we shall be more interested in its Eilenberg-Moore category. This category also has a particularly simple presentation.

Definition 4.5. (Pointed assemblies) (i) A *pointed assembly* is a pair $(X, \bot_X)$ where $\bot_X \in |X|$ such that there exists $\alpha : LX \to X$ satisfying $\alpha(\bot) = \bot_X$ and, for all $x \in |X|$, $\alpha(\ulcorner x \urcorner) = x$.

(ii) For pointed assemblies $(X, \bot_X)$ and $(Y, \bot_Y)$, a *strict* morphism $f : (X, \bot_X) \to (Y, \bot_Y)$ is a morphism $f : X \to Y$ such that $f(\bot_X) = \bot_Y$.

We write $\mathbf{Ass}_\bot$ for the category of pointed assemblies and strict maps. Note that for any pointed assembly $(X, \bot_X)$ the associated $\alpha : LX \to X$ is uniquely determined, and conversely any $\alpha : LX \to X$ satisfying $\alpha(\ulcorner x \urcorner) = x$ determines a unique pointed structure.

Proposition 4.6. The Eilenberg-Moore category of (L, η, μ) on **Ass** is isomorphic to $\mathbf{Ass}_\bot$.

PROOF In one direction, it is trivial that an algebra $\alpha : LX \to X$, for the monad gives rise to the pointed assembly $(X, \alpha(\bot))$. Conversely, if $(X, \bot_X)$ is a pointed assembly then it is easily shown that the associated $\alpha : LX \to X$ is an algebra for the monad. It is routine to check that these constructions are mutually inverse and that strict maps are in one-to-one correspondence with algebra homomorphisms. □

Note that, for any X, we have that $(LX, \bot)$ is pointed by $\mu : LLX \to LX$. Pointed objects also satisfy some basic closure properties.

Proposition 4.7.
(i) If $(X, \bot_X)$ and $(Y, \bot_Y)$ are pointed then so is $(X \times Y,\ (\bot_X, \bot_Y))$.
(ii) If $(X, \bot_X)$ is pointed then $(X^Y,\ \lambda y. \bot_X)$ is pointed for any assembly Y.

PROOF Given the correspondence of Proposition 4.6, statement (i) follows from Exercise 2 on p. 138 of (Mac Lane 1971), and statement (ii) is Lemma 3.7 of (Mulry 1992) (which uses the strength of the monad). □

The object $L\mathbf{1}$ will be of crucial importance in the sequel. It will be convenient to work with an alternative description. We write Σ for the modest set

$$|\Sigma| = \{\top, \bot\} \qquad \|\top\| \ = \ \{a \mid a\mathbf{i} = \mathbf{i}\} \qquad \|\bot\| \ = \ \{a \mid a\mathbf{i}{\downarrow} \Rightarrow \ a\mathbf{i} \in D\}.$$

It is easily seen that Σ is isomorphic to $L\mathbf{1}$. One can think of Σ as an object of truth values, with $\top$ as true and $\bot$ as false. The operation of conjunction exists as a morphism from $\Sigma \times \Sigma$ to Σ, tracked by the element $\lambda^* pa.\, \mathit{fst} p\, \mathbf{i}\, (\mathit{snd} p\, \mathbf{i})$, which we henceforth refer to as *and*. However, in general there are no morphisms representing negation, implication or disjunction. We write $\mu : L\Sigma \to \Sigma$ for the morphism satisfying $\mu(\bot) = \mu(\ulcorner \bot \urcorner) = \bot$ and $\mu(\ulcorner \top \urcorner) = \top$ (corresponding to $\mu : LL\mathbf{1} \to L\mathbf{1}$); and we write $\chi_X : LX \to \Sigma$ for the morphism satisfying $\chi_X(\bot) = \bot$ and $\chi_X(\ulcorner x \urcorner) = \top$ (corresponding to $L! : LX \to L\mathbf{1}$).

To conclude this section, we remark that the notion of partiality could be dealt with in greater generality using Rosolini's theory of *dominances* (Rosolini 1986). This gives a categorical account of how an object like Σ above can be used to specify a notion of partial map and associated lift functor. What this amounts to in the case of **Ass** is treated in detail in the first author's thesis, (Longley 1995, Ch. 4), where it is shown that for any divergence the associated Σ is a dominance. However, there are many dominances that do not arise from divergences. Despite the greater generality of dominances, we prefer to work with divergences in this paper for three reasons. First, the definition of divergence has a strong computational character. Second, the account of partiality arising from a dominance is rather more complicated than that presented above, cf. (Longley 1995, Ch. 4). Finally, the notion of divergence includes all the examples of interest that we shall treat in Section 8. However, we should mention that the forthcoming development in

Sections 5, 6, 7 and 9 can all be carried through for an arbitrary dominance, and was indeed treated at such a level of generality in (Longley 1995).

5. Completeness

In this section we begin our task of isolating an appropriate full subcategory of predomain-like objects. For this, we work in the setting of an arbitrary PCA A equipped with a divergence D. The idea is to use the derived lift functor L as the analogue of the lift functor of classical domain theory. Thus, for example, Σ will play the rôle of the classical Sierpinski space.

As in classical domain theory, we shall require a predomain to satisfy a form of chain completeness. A first difference, however, is that we shall not assume that predomains are partially ordered (at least not until Section 9), thus we need a notion of "ascending chain" that does not presuppose a partial order.

As Σ is our analogue of Sierpinski space, a natural notion of two-element "ascending chain" is given by morphisms from Σ to X. In other words, one may think of Σ as a *generic* two-element chain. More generally, $L^n\mathbf{1}$ provides a notion of generic $(n+1)$-element chain. Our notion of infinite ascending chain will be obtained similarly, using an object acting as the generic such chain. The object, I, we use for this purpose will carry an initial algebra structure for the lift functor. This is intuitively reasonable as one would expect the generic infinite chain to be obtained by "freely iterating" the lift functor.

We take a similar approach to definining the notion of the "supremum" of an ascending chain. For this purpose we construct an object F which acts as a generic ascending chain together with its supremum. Intuitively, F is the "chain completion" of I. Categorically, it will be defined as the final coalgebra of the lift functor.

We begin by constructing I and F. For $i \in \mathbb{N} \cup \{\infty\}$ define $p_i : N \to \Sigma$ by: $p_i(n) = \top$ if $n < i$, and $p_i(n) = \bot$ if $n \geq i$. It is easy to find realizers for the p_i.

Definition 5.1. **(I and F)** Let F be the assembly defined by

$$|F| \;=\; \mathbb{N} \cup \{\infty\}, \qquad \|i\|_F \;=\; \|p_i\|_{\Sigma^N}.$$

Let I be the regular subobject of F determined by $|I| = \mathbb{N}$. We write $\iota : I \to F$ for the inclusion tracked by $\mathbf{i}$.

We observe that F is a retract of Σ^N. The inclusion from F to Σ^N is obvious. The retraction is given by the function $f : |\Sigma^N| \to |F|$ defined by:

$$f(c)(n) \;=\; \begin{cases} \top & \text{if, for all } m \text{ with } 0 \leq m \leq n,\ c(m) = \top, \\ \bot & \text{otherwise.} \end{cases}$$

This is tracked by $\lambda^* cn.\ rec\,(c\,\overline{0})\;(\lambda^* mp.\ and\,p\;(c\;(succ\,m)))\;n$.

Let $\sigma_F : |LF| \to |F|$ be the function defined by $\sigma(\bot) = 0$, $\sigma(\ulcorner n \urcorner) = n{+}1$, $\sigma(\ulcorner \infty \urcorner) = \infty$. Clearly this restricts to $\sigma_I : |LI| \to |I|$. Then both σ_F and σ_I are isomorphisms in **Ass**, as each is tracked by $\lambda^* qm.\ \mathit{if}\,(\mathit{iszero}\ m)(\mathit{fst}(q\mathbf{i}))(\mathit{snd}(q\mathbf{i})\,(\mathit{pred}\ m))$ with its inverse tracked by $\lambda^* pa.\ \langle p\,\overline{0}\,\mathbf{i},\ \lambda^* m.\,p\,(succ\,m)\rangle$. We write τ_F and τ_I for the inverses. We can now state the universal properties of I and F.

Proposition 5.2.
(i) For any $f : LX \to X$ there is a unique $c : I \to X$ such that diagram (1) below commutes.
(ii) For any $g : Y \to LY$ there is a unique $h : Y \to F$ such that diagram (2) below commutes.

$$
\begin{array}{ccc}
LI & \xrightarrow{\;Lc\;} & LX \\
{\scriptstyle \sigma_I}\downarrow & (1) & \downarrow{\scriptstyle f} \\
I & \xrightarrow{\;c\;} & X
\end{array}
\qquad\qquad
\begin{array}{ccc}
LY & \xrightarrow{\;Lh\;} & LF \\
{\scriptstyle g}\uparrow & (2) & \uparrow{\scriptstyle \tau_F} \\
Y & \xrightarrow{\;h\;} & F
\end{array}
$$

PROOF (i) It is easy to check that $c : |I| \to |X|$ makes diagram (1) commute if and only if it satisfies: $c(0) = f(\bot)$ and $c(n+1) = f(\ulcorner c(n) \urcorner)$. So we need only show that the unique c determined by these equations is tracked. Suppose t tracks f. Consider any $r \in A$ satisfying

$$rx \quad \simeq \quad t(\lambda^* a.\, \langle x\, \overline{0}\, \mathbf{i},\, r(\lambda^* m.\, x\, (succ\, m))\rangle).$$

Then, by a straightforward induction on n, one shows that $p \in \|n\|_I$ implies $(r\,p) \in \|f(n)\|$. Therefore f is tracked by $\mathbf{z}(\lambda^* r\, p.\, t(\lambda^* a.\, \langle p\, \overline{0}\, \mathbf{i},\, r(\lambda^* m.\, p(succ\, m))\rangle))$.

(ii) Define $\overline{g} = \mu \circ Lg : LY \to LY$. Let t be any element of A tracking $\overline{g}$. It is easy to check that the only $h : |Y| \to |F|$ that can make diagram (2) commute must be defined by:

$$h(y) \quad = \quad \begin{cases} n & \text{if } \overline{g}^{\,n}(\ulcorner y \urcorner) \neq \bot \text{ and } \overline{g}^{\,n+1}(\ulcorner y \urcorner) = \bot, \\ \infty & \text{if, for all } n,\, \overline{g}^{\,n}(\ulcorner y \urcorner) \neq \bot. \end{cases}$$

We need only show that h is tracked. First we note that $f : |Y| \to |Y^N|$ defined by $f(y)(n) = \overline{g}^{\,n}(\ulcorner y \urcorner)$ is tracked by $r = \lambda^* y\, n.\, rec\, (\lambda^* a.\, \langle \mathbf{i},\, y\rangle)\, (\lambda^* m\, z.\, t(z))\, n$. Therefore h is tracked by $\lambda^* y\, n\, a.\, fst\, (r\; y\; succ(n)\; \mathbf{i})$. This completes the proof. □

Statement (i) says that σ_I is the initial algebra of the lift functor, and statement (ii) says that τ_F is the final coalgebra. Note that, by construction, both I and F are modest, so their universal properties hold also in **Mod**.

There are some useful morphisms associated with F and I. First, we note the "shift" map $up : F \to F$ defined so that $up(n) = n+1$ and $up(\infty) = \infty$. This is tracked by $\lambda^* p\, m\, a.if\, (iszero\, m)\, (\lambda^* a.\, \mathbf{i})\, (p\, (pred\, m))$. Clearly it restricts to a map $up : I \to I$. Also, note the map $min : F \times F \to F$ such that $min(i, j)$ is the evident minimum of i and j, which is realized by $\lambda^* p\, q\, n.\, and\, (p\, n)\, (q\, n)$. Note that min is commutative, and restricts to a map $min : I \times F \to I$. Incidentally, the above maps can all be defined abstractly just using the universal properties of I and F together with the primitive operations of the lift monad.

We now give a useful consequence of the initial-algebra property of I. We show that an endomorphism $f : X \to X$ generates a canonical chain in X whenever X carries a pointed structure. First we observe that $(I, 0)$ is a pointed assembly, with the required map from LI to I given by $\sigma \circ \mu \circ L\tau$.

Proposition 5.3. For any endomorphism $f : X \to X$ and any pointed structure $\bot_X$ on

X there is a unique strict $c : (I, 0) \to (X, \perp_X)$ such that the diagram below commutes.

$$\begin{array}{ccc} I & \xrightarrow{\ c\ } & X \\ {\scriptstyle up}\downarrow & (3) & \downarrow{\scriptstyle f} \\ I & \xrightarrow{\ c\ } & X \end{array}$$

PROOF Let $\alpha : LX \to X$ be the map giving the pointed structure $(X, \perp_X)$. It is easily checked that any morphism $c : I \to X$ is both strict (from $(I, 0)$ to $(X, \perp_X)$) and makes diagram (3) above commute if and only if $f \circ \alpha \circ Lc = c \circ \sigma_I$. So, by Proposition 5.2, there is indeed a unique strict c making diagram (3) commute. □

In fact the property stated in the above proposition is equivalent to the initial-algebra property of I. Indeed such an equivalence can be formulated for an arbitrary monad (see Theorem A.5 of (Joyal and Moerdijk 1995), due to Bénabou and Jibladze).

As motivated earlier, we are regarding morphisms $I \to X$ as ascending chains in X, and morphisms $F \to X$ as ascending chains with "suprema". This suggests that we consider an object X to be "chain-complete" if every morphism $c : I \to X$ extends to a unique morphism $\overline{c} : F \to X$. In fact, we shall require this property to hold in a "uniform" way, in the sense that there is a morphism that computes $\overline{c}$ from c.

Given a morphism $f : X \to Y$ and an object Z, we write $Z^f : Z^Y \to Z^X$ for the induced morphism mapping any $g : Y \to Z$ to $g \circ f : X \to Z$. We say that Z is *orthogonal* to f if Z^f is an isomorphism.

Definition 5.4. (Complete object) An assembly X is said to be *complete* if X is orthogonal to the inclusion $\iota : I \to F$.

Given $c : I \to X$ where X is complete, we write $\overline{c}$ for the unique morphism from F to X extending c. Similarly we write $\overline{(\cdot)} : X^I \to X^F$ for the inverse to X^ι. Note that given $f : I \to X$ and $g : X \to Y$ where X and Y are complete it holds that $g \circ \overline{f} = \overline{g \circ f} : F \to Y$.

We now give a straightforward alternative characterization of completeness. Completeness requires two properties of X: first, that every chain in $I \to X$ has at most one extension to a morphism from F to X; second, that every chain has at least one such extension and there is a morphism $X^I \to X^F$ finding extensions. The second condition turns out to be equivalent to the existence of a morphism $\bigsqcup : X^I \to X$ satisfying a simple property ensuring it finds the "suprema" of chains.

We say that a chain $c : I \to X$ is *eventually constant* if for some m it holds that $c(n) = c(m)$ for all $n \geq m$. We say that $c(m)$ is the *eventual value* of such an eventually constant chain. An *eventual-value operator* on X is a map $\bigsqcup : X^I \to X$ such that, for any eventually constant chain c in X, it holds that $\bigsqcup c$ is the eventual value of c.

Proposition 5.5. X is complete if and only if the following two conditions hold:
(i) for any $c_1, c_2 : F \to X$ that restrict to the same I-chain, $c_1 = c_2$;
(ii) X has an eventual-value operator.

PROOF For the left-to-right direction, if X is complete then X^ι is mono so (i) is satisfied.

For (ii), it is easy to see that there is a morphism $\bigsqcup_X : X^I \to X$ representing the function $\bigsqcup_X(c) = \overline{c}(\infty)$. Suppose that c is an eventually-constant chain that becomes constant at m. We must show that $\bigsqcup_X(c) = c(m)$. But $\bigsqcup_X(c) = \overline{c}(\infty) = \overline{c}(up^m(\infty)) = \overline{c \circ up^m}(\infty)$, where the last equality holds because both $\overline{c} \circ up^m$ and $\overline{c \circ up^m}$ from F to X restrict to the same morphism from I to X. But, as c becomes constant at m, we have that $\lambda i.\, c(m) : F \to X$ extends $c \circ up^m : I \to X$. Therefore $\overline{c \circ up^m} = \lambda i.\, c(m)$, thus indeed $\bigsqcup_X(c) = c(m)$.

For the converse, suppose that (i) and (ii) hold. We must find an inverse to $X^\iota : X^F \to X^I$. Let $\bigsqcup$ be an eventual-value operator. Define $comp : X^I \to X^F$ to be the composite

$$X^I \xrightarrow{X^{min}} X^{I \times F} \xrightarrow{\cong} (X^I)^F \xrightarrow{\bigsqcup^F} X^F.$$

Thus $comp(c)(i) = \bigsqcup(\lambda n.\, c(min(n, i)))$. By (i), X^ι is mono, so it suffices to prove $X^\iota \circ comp = \mathrm{id}$. But given $c \in X^i$ and $m \in I$ we have $X^\iota(comp(c))(m) = comp(c)(m) = \bigsqcup(\lambda n.\, c(min(n, m)))$, and $\lambda n.\, c(min(n, m))$ is clearly an eventually constant chain with eventual value $c(m)$. Thus $\bigsqcup(\lambda n.\, c(min(n, m))) = c(m)$. So $X^\iota(comp(c)) = c$ as required. □

Next we give some consequences of completeness. For (iv) below, we say that a morphism $q : I \to I$ is *monotonic* if $m \leq n$ implies $q(m) \leq q(n)$; and we say it is *cofinal* if, for all n, there exists m such that $q(m) \geq n$.

Theorem 5.6. If X and Y are complete then the following statements hold:
(i) X is modest.
(ii) X has a unique eventual-value operator $\bigsqcup_X$.
(iii) Any morphism f from X to Y is continuous, i.e., for all $c \in X^I$, $f(\bigsqcup_X c) = \bigsqcup_Y(f \circ c)$.
(iv) For any monotonic cofinal $q : I \to I$ and any $c \in X^I$, it holds that $\bigsqcup_X(c \circ q) = \bigsqcup_X c$.
(v) X carries at most one pointed structure.

Proof (i) Suppose X is complete and we have $x, y \in |X|$ with $a \in \|x\| \cap \|y\|$. Consider $c_1, c_2 : F \to X$ defined by: $c_1(n) = c_2(n) = x$ for $n \in |I|$; $c_1(\infty) = x$ and $c_2(\infty) = y$. Clearly both are tracked by $\lambda^* z.\, a$. Then both c_1 and c_2 restrict to the same chain $I \to X$. So, by completeness, $c_1 = c_2$ hence $x = c_1(\infty) = c_2(\infty) = y$. Thus X is indeed modest.

(ii) Let $\bigsqcup$ be an eventual-value operator on X. As in the proof of Proposition 5.5 the morphism $comp : X^I \to X^F$ is the inverse of $X^\iota : X^F \to X^I$. Thus $comp(c) = \overline{c}$. Then $\bigsqcup(c) = \bigsqcup(\lambda n.\, c(min(n, \infty))) = comp(c)(\infty) = \overline{c}(\infty)$. Thus $\bigsqcup$ is necessarily the eventual-value operator $\bigsqcup_X$ defined in the proof of Proposition 5.5.

(iii) We have that $f(\bigsqcup_X c) = f(\overline{c}(\infty)) = \overline{f \circ c}(\infty) = \bigsqcup_Y(f \circ c)$.

(iv) This follows from (ii) because $c \mapsto \bigsqcup_X(c \circ q)$ is clearly an eventual-value operator.

(v) Suppose X carries pointed structures $(X, \bot_1)$ and $(X, \bot_2)$. Then $(X \times X, (\bot_1, \bot_2))$ is pointed by Proposition 4.7(i). Consider the morphism $\langle \pi_2, \pi_1 \rangle : (X \times X) \to (X \times X)$. By Proposition 5.3 there is a unique strict $c : (I, 0) \to (X \times X, (\bot_1, \bot_2))$ such that $\langle \pi_2, \pi_1 \rangle \circ c = c \circ up$. It follows that, for even n, $c(n) = (\bot_1, \bot_2)$ and, for odd n, $c(n) = (\bot_2, \bot_1)$. Next consider the evident morphisms $q_1, q_2 : I \to I$ such that $q_1(n) = 2n$ and $q_2(n) = 2n + 1$, which are both monotonic and cofinal. Thus, by (iv), we have $\bigsqcup_X(\pi_1 \circ c) = \bigsqcup_X(\pi_1 \circ c \circ q_1) = \bigsqcup_X(\lambda n.\, \bot_1) = \bot_1$, and similarly $\bigsqcup_X(\pi_1 \circ c) = \bigsqcup_X(\pi_1 \circ c \circ q_2) = \bigsqcup_X(\lambda n.\, \bot_2) = \bot_2$. Thus indeed $\bot_1 = \bot_2$. □

The proof above demonstrates some of the interesting properties of the operator $\bigsqcup_X$. The operator also satisfies the equational axioms of a *formal lub operator* identified by Fiore and Plotkin in their (as yet unpublished) work on axiomatic domain theory. However, the properties highlighted above suffice for our purposes.

Next we consider closure properties of categories of complete objects. We write **C** for the full subcategory of **Ass** (indeed of **Mod**) consisting of the complete objects.

Theorem 5.7. The category **C** is cartesian closed with finite limits and the inclusion in **Ass** preserves this structure. Furthermore, **C** is an exponential ideal of **Ass**.

PROOF Suppose X and Y are complete, then the composite:

$$(X \times Y)^I \xrightarrow{\cong} X^I \times Y^I \xrightarrow{\overline{(\cdot)} \times \overline{(\cdot)}} X^F \times Y^F \xrightarrow{\cong} (X \times Y)^F$$

gives the inverse to $(X \times Y)^\iota$. Thus $X \times Y$ is complete.

For the exponential ideal suppose X is complete and Y is arbitrary. Then the composite

$$(X^Y)^I \xrightarrow{\cong} (X^I)^Y \xrightarrow{\overline{(\cdot)}^Y} (X^F)^Y \xrightarrow{\cong} (X^Y)^F$$

gives the inverse to $(X^Y)^\iota$. Thus X^Y is indeed complete. Cartesian closure follows.

It remains to check that the equalizer of two maps between complete objects is complete. Let $e : Z \to X$ be the equalizer of $f, g : X \rightrightarrows Y$ as defined in the proof of Theorem 3.2. Given $c : I \to Z$ we have $f \circ \overline{e \circ c} = \overline{f \circ e \circ c} = \overline{g \circ e \circ c} = g \circ \overline{e \circ c}$. Therefore $\overline{e \circ c} : F \to X$ factors as $e \circ \overline{c}$, where it easy to see that $\overline{c} : F \to Z$ is the unique extension of c. Moreover the function mapping $c \in |Z^I|$ to $\overline{c} \in |Z^F|$ is clearly tracked by the same $r \in A$ that tracks $\overline{(\cdot)} : X^I \to X^F$. It follows that Z is complete. □

Thus **C** has good "limiting" structure. A useful category of predomains should also have coproducts, a natural number object, be closed under lifting and have a fixpoint object. The next two sections will be devoted to obtaining a category with such properties.

6. The completeness axiom

In this section we investigate a property which, when it holds, enables us to proceed a long way towards obtaining a suitable category of predomains. We formulate this property as an "axiom":

Axiom 1: Σ is complete.

This axiom will be referred to as the *completeness axiom*. Although there exist PCAs and dominances such that the axiom does not hold (an example will be given later) these are pathological cases. For a whole variety of natural choices of PCAs and dominances the axiom does hold, as the examples of Section 8 will testify.†

In the next two sections we shall analyse the many consequences of the completeness axiom in some detail. First, however, we give a reformulation of the axiom which both

† The reader familiar with (Hyland 1990) is warned that our Axiom 1 is not the same as Axiom 9 of that paper. This is because the object ω of *loc. cit.* is *not* the initial algebra of the lift functor.

illuminates its nature and provides useful conditions for showing whether or not the axiom is satisfied in examples.

Proposition 6.1. Σ is complete if and only if the following three conditions hold:
(i) There is no morphism $neg : \Sigma \to \Sigma$ such that $neg(\top) = \bot$ and $neg(\bot) = \top$.
(ii) There is no morphism $inf : F \to \Sigma$ such that, $inf(\infty) = \top$ and, for all n, $inf(n) = \bot$.
(iii) There is a morphism $\bigsqcup : \Sigma^I \to \Sigma$ such that $\bigsqcup(p) = \top$ if there exists n with $p(n) = \top$, and $\bigsqcup(p) = \bot$ otherwise.

To prove the proposition, we begin by analysing the notion of morphism out of Σ qua two element ascending chain.

Definition 6.2. (Path relation) For $x, y \in |X|$ we write $x \rightsquigarrow y$ if there exists $l : \Sigma \to X$ such that $l(\bot) = x$ and $l(\top) = y$.

The path relation is called the *link* relation in (Phoa 1990). We adopt the notation and terminology of (Fiore 1995). Although (obviously) reflexive, the path relation is not in general transitive.

An important trivial observation is that the path relation is preserved by arbitrary morphisms, i.e. if $x \rightsquigarrow y$ in X, and $f : X \to Y$ then $f(x) \rightsquigarrow f(y)$. It turns out that, when condition (i) of Proposition 6.1 holds, the path relation gives a partial order on the objects Σ, I, F and $L\mathbf{2}$. It follows that that any morphism between these objects is necessarily monotonic.

First we specify the partial orders that arise. We define the *natural* order on Σ, I, F and $L\mathbf{2}$ as follows: for Σ it is $\bot \leq \top$; for I and F it is the expected ascending order (equivalently the pointwise order on I and F as subobjects of Σ^N); for $L\mathbf{2}$ it is the order $\ulcorner tt \urcorner \geq \bot \leq \ulcorner ff \urcorner$.

Lemma 6.3. If condition (i) of Proposition 6.1 holds then the natural orders on Σ, I, F and $L\mathbf{2}$ coincide with the path relation.

Proof A tedious, but easy, exhaustive analysis of the possibilities: for each x, y such that $x \leq y$ in the natural order on X (where X is one of Σ, I, F and $L\mathbf{2}$) one finds an appropriate $l : \Sigma \to X$ showing that $x \rightsquigarrow y$; conversely, for each x, y such that $x \not\leq y$ one finds $p : X \to \Sigma$ such that $p(x) = \top$ and $p(y) = \bot$ showing that $x \not\rightsquigarrow y$ by condition (i). □

Thus, when condition (i) holds, all morphisms between Σ, I, F and $L\mathbf{2}$ are monotonic with respect to the natural order.

We are now in a position to prove Proposition 6.1. For the left-to-right direction, suppose that Σ is complete. We show that (i)–(iii) hold.

(i) If $neg : \Sigma \to \Sigma$ existed then $neg \circ \mu \circ L(neg) : L\Sigma \to \Sigma$ would display $(\Sigma, \top)$ as a pointed structure, contradicting Theorem 5.6(v).

(ii) If $inf : F \to \Sigma$ existed then it would restrict to the same chain as $(\lambda i. \bot) : F \to \Sigma$. But this chain must have a unique extension.

(iii) By Lemma 6.3 and the consequent monotonicity of morphisms, every chain in Σ is eventually constant and the eventual-value operator $\bigsqcup_\Sigma$ has the required properties.

For the converse, suppose that (i)–(iii) hold. It follows from Lemma 6.3 and (ii) that, for any $d : F \to \Sigma$, $d(\infty) = \top$ iff there exists $n \in |I|$ such that $d(n) = \top$. Thus every

$d : F \to \Sigma$ is determined by its restriction to I. Further, it is clear that any morphism satisfying the conditions of (iii) is an eventual-value operator. Therefore, by Proposition 5.5, Σ is indeed complete. This concludes the proof of Proposition 6.1.

As a first use of Proposition 6.1, we give an example PCA and divergence such that Axiom 1 fails. (Many situations in which Axiom 1 holds will be discussed in Section 8.) In (Plotkin 1995), Plotkin constructs a *finitely separable* model of the untyped λ-calculus. This provides a total PCA, A, such that for any distinct $a, b \in A$ there exists $r \in A$ such that $ra = \mathit{true}$ and $rb = \mathit{false}$. If one takes the divergence $\{\mathbf{kz}\}$ in this PCA, it is easily seen that condition (i) of Proposition 6.1 is violated.

Next we turn to consequences of Axiom 1. First, as Σ is complete and F is a retract of Σ^N, it follows from Theorem 5.7 that F is complete. More strikingly:

Theorem 6.4. If Axiom 1 holds then **C** has finite coproducts and nno and the inclusion in **Ass** preserves this structure.

PROOF The initial object is trivially complete. For binary coproducts suppose that X and Y are complete. First we show that the image of any $c : I \to X+Y$ lies either entirely in X or entirely in Y. Suppose, for contradiction, that $c(m) = (0, x)$ and $c(n) = (1, y)$ where, without loss of generality, $m \leq n$. Then the composite

$$I \xrightarrow{c} X + Y \xrightarrow{!+!} 1 + 1 \xrightarrow{[\top,\bot]} \Sigma$$

gives a non-monotonic chain in Σ, contradicting Axiom 1. Next we show that $(X+Y)^I \cong X^I + Y^I$. The evident function from right-to-left exists in general. Its inverse separates chains in $(X + Y)^I$ according to whether they lie entirely in X or entirely in Y. It is tracked by $\lambda^* c.\, \mathit{if}\,(\mathit{fst}(ci_o))\,\langle \mathit{true},\ \lambda^* e.\, \mathit{snd}(ce)\rangle\,\langle \mathit{false},\ \lambda^* e.\, \mathit{snd}(ce)\rangle$ where $i_0 \in \|0\|_I$. A similar argument shows that $(X + Y)^F \cong X^F + Y^F$. Finally the composite:

$$(X+Y)^I \xrightarrow{\cong} X^I + Y^I \xrightarrow{\overline{(\cdot)} + \overline{(\cdot)}} X^F + Y^F \xrightarrow{\cong} (X+Y)^F$$

gives the inverse to $(X + Y)^\iota$. Thus $X + Y$ is complete.

For the nno, one shows, as above, that every chain in $I \to N$ is constant, and similarly for $F \to N$. Indeed $N^I \cong N \cong N^F$, giving the required inverse to N^ι. □

However, we have still not shown that **C** is closed under lifting. In fact we do not believe that this holds in general (although we lack a counterexample, see the discussion at the end of Section 7).

The apparent failure of closure under lifting means that **C** is deficient as a category of predomains. In the next section we remedy this deficiency by isolating a subcategory of **C** that does turn out to be closed under lifting. Beforehand, we conclude this section by showing that the subcategory of *pointed* complete objects does at least provide a reasonable category of *domains*.

We have seen (Theorem 5.6(v)) that a complete object X can have at most one point $\bot_X$, so we shall refer to a complete pointed object $(X, \bot_X)$ simply as X. Let **D** be the *full* subcategory of **C** consisting of the complete pointed objects (thus we are *not* requiring the maps in **D** to be strict). By Proposition 4.7 and Theorem 5.7, **D** is cartesian-closed (indeed it is an exponential ideal of **Ass**). We now show that it is closed under lifting and that every endomorphism has a canonical fixed-point.

Proposition 6.5. If Axiom 1 holds then **D** is closed under lifting.

PROOF Suppose X is complete and pointed. We show that LX is a retract of $X \times \Sigma$, whence, by Axiom 1 and Theorem 5.7, LX is complete (it is trivially pointed). Let $\alpha : LX \to X$ be the map giving the pointed structure. The embedding of the retract is given by $\langle \alpha, \chi \rangle : LX \to X \times \Sigma$. The associated retraction is $f : |X \times \Sigma| \to |LX|$ defined by $f(x, \bot) = \bot$ and $f(x, \top) = \ulcorner x \urcorner$, which is tracked by $\lambda^* p\, a.\, \langle \mathit{snd}\, p\, \mathbf{i}, \mathit{fst}\, p \rangle$. □

Proposition 6.6. Any $f : X \to X$, where X is in **D**, has a fixed-point.

PROOF Let $c : I \to X$ be the unique strict map such that $f \circ c = c \circ \mathit{up}$ (given by Proposition 5.3). The fixpoint is $\bigsqcup_X c$, because $f(\bigsqcup_X c) = \bigsqcup_X (f \circ c) = \bigsqcup_X (c \circ \mathit{up}) = \bigsqcup_X c$ (the last equation by Theorem 5.6(iv)). □

It is straightforward to proceed from here to establish fixed-point operators $Y_X : X^X \to X$ in **D**, characterized as the unique such operators satisfying the analogue of Plotkin's property of *uniformity*, see (Mulry 1992; Simpson 1992).

7. A category of predomains

In this section we define our category of predomains, the category of *well-complete* objects. The benefit over the complete objects is that we can prove that the well-complete objects are closed under lifting. However, in order to retain closure under coproducts it will be necessary to introduce another axiom.

Definition 7.1. (Well-complete object) An assembly X is *well-complete* if LX is complete.

We write **WC** for the full subcategory of **Ass** consisting of well-complete objects.

Theorem 7.2. The category **WC** has non-empty finite limits and its inclusion in **Ass** preserves them. Further, if Axiom 1 holds then **WC**: is a full subcategory of **C**; is closed under lifting; and is an exponential ideal of **Ass**.

PROOF For binary products, suppose LX and LY are complete. We show that $L(X \times Y)$ is a retract of $LX \times LY$, thus indeed $L(X \times Y)$ is complete, by Theorem 5.7. The embedding is $\langle L\pi_1, L\pi_2 \rangle : L(X \times Y) \to (LX \times LY)$. Its associated retraction is $f : |(LX \times LY)| \to |L(X \times Y)|$ defined by $f(\bot, z) = f(z, \bot) = \bot$ and $f(\ulcorner x \urcorner, \ulcorner y \urcorner) = \ulcorner (x, y) \urcorner$, which is easily shown to be tracked.

For equalizers, suppose $e : Z \to X$ is the equalizer of $f, g : X \to Y$ and LX and LY are complete. By the explicit construction of equalizers (in the proof of Theorem 3.2) and the explicit definition of lifting, it is immediate that L preserves equalizer diagrams. Thus Le equalizes Lf and Lg. So, by Theorem 5.7, LZ is complete as required.

For the remainder of the proof we assume that Axiom 1 holds.

To show X well-complete implies X complete, assume LX is complete. It is easily checked that $\eta : X \to LX$ equalizes $\chi : LX \to \Sigma$ and $(\lambda z.\top) : LX \to \Sigma$. Hence X is complete by Theorem 5.7.

For closure under lifting we must show that LX complete implies LLX complete. But this follows from Proposition 6.5.

It remains to show that **WC** is an exponential ideal of **Ass**. Suppose LX is complete, we must show that X^Y is well-complete for an arbitrary assembly Y. Observe that X^Y

is the equalizer of $\chi^Y : (LX)^Y \to \Sigma^Y$ and $(f \mapsto \lambda y.\top) : (LX)^Y \to \Sigma^Y$. By Theorem 5.7 both $(LX)^Y$ and Σ^Y are complete. But, by Proposition 4.7(ii), both are pointed and hence, by Proposition 6.5, well-complete. Thus X^Y is indeed well-complete by the closure of **WC** under equalizers. □

The theorem shows that, in the presence of Axiom 1, **WC** has reasonable closure properties. First observe that it trivially contains the terminal object (and thus has all finite limits). Further Σ is well-complete (as it is isomorphic to $L1$), and hence so is F (by the same argument as for **C**). Also, **WC** is cartesian closed (because it is an exponential ideal). Moreover, it follows from the definition of well-completeness that **WC** is characterized as the largest full subcategory of **C** that is closed under lifting. Actually, it is most embarrassing that we do not at present have an example PCA and divergence for which Axiom 1 holds but **C** and **WC** differ. Nonetheless, the benefit of working with well-completeness is that we can prove that **WC** has the properties we desire.

Proposition 7.3. If Axiom 1 holds then $\sigma_F : LF \to F$ is the initial algebra for the lift functor in **WC**.

PROOF Given $f : LX \to X$ (where X is well-complete) we must show that there is a unique d making the diagram below commute.

$$\begin{array}{ccc} LF & \xrightarrow{\;Ld\;} & LX \\ \sigma_F \big\downarrow & (4) & \big\downarrow f \\ I & \xrightarrow{\;d\;} & X \end{array}$$

Let $c : I \to X$ be the unique morphism making diagram (1) of Proposition 5.2 commute. It is easily checked that diagram (4) commutes when d is $\overline{c}$, and also that any other d making it commute restricts to c. Thus, because X is complete, $\overline{c}$ is indeed the unique such d. □

Note that only the completeness of X was used in the above proof. However, the same proposition cannot be stated for **C** because we do not know that L is an endofunctor on **C**.

By Proposition 5.2(ii), σ_F^{-1} is the final coalgebra of the lift functor in **WC**. Together with the above proposition this implies that F is a *fixpoint object* in **WC**, see (Crole and Pitts 1992, Lemma 2.3). The many consequences of the presence of a fixpoint object are discussed in (Crole and Pitts 1992; Mulry 1992; Simpson 1992).

We still have not shown that **WC** is closed under finite coproducts in **Ass** and contains the natural number object N. In fact Axiom 1 does not appear to be sufficiently strong to prove this. To address this problem we introduce another axiom.

Axiom 2: $L\mathbf{2}$ is complete.

As Σ is a retract of $L\mathbf{2}$, it follows from Theorem 7.2 that Axiom 2 implies Axiom 1. Thus we shall refer to Axiom 2 as the *strong completeness axiom.* We do not believe that the converse implication holds, but unfortunately we do not have a counterexample (see the discussion at the end of the section).

The following analogue of Proposition 6.1 is helpful for checking that the strong completeness axiom holds in examples.

Proposition 7.4. $L\mathbf{2}$ is complete if and only if (i) and (ii) of Proposition 6.1 hold together with:
(iii′) there is a morphism $\bigsqcup : (L\mathbf{2})^I \to L\mathbf{2}$ such that: $\bigsqcup(p) = \ulcorner tt \urcorner$ if there exists n with $p(n) = \ulcorner tt \urcorner$; $\bigsqcup(p) = \ulcorner ff \urcorner$ if there exists n with $p(n) = \ulcorner ff \urcorner$; and $\bigsqcup(p) = \bot$ otherwise.

PROOF If $L\mathbf{2}$ is complete, then Σ is complete so (i) and (ii) hold by Proposition 6.1. For (iii′) we have by Lemma 6.3 that every chain in $L\mathbf{2}$ is eventually constant and the eventual-value operator $\bigsqcup_{L\mathbf{2}}$ has the required properties.

Conversely, if (i) and (ii) hold then every $d : F \to L\mathbf{2}$ is determined by its restriction to I. Moreover, any morphism satisfying (iii′) is an eventual-value operator. So $L\mathbf{2}$ is complete by Proposition 5.5. □

The next theorem shows that Axiom 2 does indeed imply that **WC** has the desired additional structure.

Theorem 7.5. If Axiom 2 holds then **WC** has finite coproducts and nno, and the inclusion in **Ass** preserves them.

PROOF The initial object is trivially well-complete. For binary coproducts, suppose X and Y are well-complete. It is straightforward to check that $X + Y$ is the equalizer of the morphisms:

$$(LX \times LY) + (LX \times LY) \underset{((\lambda z.\top) \times (\lambda z.\bot)) + ((\lambda z.\bot) \times (\lambda z.\top))}{\overset{(\chi \times \chi) + (\chi \times \chi)}{\rightrightarrows}} (\Sigma \times \Sigma) + (\Sigma \times \Sigma)$$

But the source is isomorphic to $\mathbf{2} \times LX \times LY$ and the target to $\mathbf{2} \times \Sigma \times \Sigma$, both of which are well-complete by Theorem 7.2. Therefore the equalizer, $X + Y$, is well-complete too.

To show that N is well-complete, consider the regular subobjects C and D of $\mathbf{2}^N$ defined by:

$$\begin{aligned} |D| &= \{d \in |\mathbf{2}^N| \mid \text{forall } n,\ d(n) = tt \text{ implies } d(n+1) = tt\}, \\ |C| &= \{d \in |D| \mid \text{there exists } n \text{ such that } d(n) = tt\}. \end{aligned}$$

We shall show that C is well-complete and that $N \cong C$, hence N is well-complete. First, D is easily exhibited as a retract of $\mathbf{2}^N$ (cf. the proof that F is a retract of Σ^N), and is therefore well-complete by Theorem 7.2. Next we define a morphism $g : D \to \Sigma$ and show that C is the equalizer of g, $(\lambda d.\top) : D \to \Sigma$, whence C is well-complete by Theorem 7.2. To construct $g : D \to \Sigma$, consider the endomorphism $f : \Sigma^D \to \Sigma^D$ defined by:

$$f(p)(d) = \begin{cases} \top & \text{if } d(0) = tt, \\ p(\lambda n.d(n+1)) & \text{otherwise,} \end{cases}$$

(it is easy to find an element of A tracking f). By Proposition 4.7(ii) Σ^D is pointed. So, by Proposition 5.3, there is a unique strict $c : I \to \Sigma^D$ such that $f \circ c = c \circ up$. We take $g : D \to \Sigma$ to be the morphism named by $\bigsqcup_{\Sigma^D} c$ (which is the canonical fixpoint of f given by Proposition 6.6). We must show that $g(d) = \top$ if and only if $d \in |C|$. If $d \in |C|$ then there exists m such that $d(m) = tt$. Then, for all $n \geq m$, $c(n)(d) = \top$. So $\bigsqcup_{\Sigma}(\lambda n.\, c(n)(d)) = \top$. Therefore $(\bigsqcup_{\Sigma^D} c)(d) = \top$ (as it is easily checked that $\bigsqcup_{\Sigma^D}$

is computed pointwise). Thus indeed $g(d) = \top$. Conversely, if $d \notin |C|$ then, for all n, $d(n) = \mathit{ff}$. So, for all n, $c(n)(d) = \bot$, whence $\bigsqcup_\Sigma(\lambda n.\, c(n)(d)) = \bot$. Then, as above, $(\bigsqcup_{\Sigma^D} c)(d) = \bot$ as required.

It remains to prove that $N \cong C$. The isomorphism from N to C maps n to the d satisfying $d(m) = \mathit{ff}$ iff $m < n$, which is easily seen to be tracked. Its inverse is the function $h : |C| \to |N|$ recursively defined by:

$$h(d) \quad = \quad \begin{cases} 0 & \text{if } d(0) = \mathit{tt}, \\ 1 + h(\lambda n.\, d(n+1)) & \text{otherwise.} \end{cases}$$

This is tracked by $\mathbf{z}(\lambda^* h\, d.\, \mathit{if}\,(d\,\overline{0})\;\overline{0}\;(\mathit{succ}(h(\lambda^* n.\, d(\mathit{succ}\, n)))))$. □

The structure we have identified on **WC** is sufficient to interpret simple type theories with recursion, such as variants of PCF (Plotkin 1977) with diverse calling mechanisms, see (Longley 1995) for details. In Section 11 we shall sketch how **WC** also supports more complicated constructions such as those required to interpret recursive and polymorphic types.

We conclude this section with a summary of the questions raised above concerning counterexamples to various implications. In the presence of Axiom 1 we were able to derive everything we wanted about **C** except closure under lifting (Theorems 5.7 and 6.4). Similarly, we were able to derive everything we wanted about **WC** except closure under coproducts in **Ass** (Theorem 7.2). Thus there are two main open questions. Firstly, does Axiom 1 imply that **C** is closed under lifting? This is equivalent to the question of whether Axiom 1 implies that completeness coincides with well-completeness. Secondly, does Axiom 1 imply that **WC** is closed under coproducts? This is equivalent to the question of whether Axiom 1 implies Axiom 2. It should be noted that a positive answer to the first question would, by Theorem 6.4, provide a positive answer to the second. However, we believe that the answer to both questions is negative. Unfortunately, none of the examples considered in the next section can be used to resolve either of the above questions, because in all cases it turns out that the concepts of completeness and well-completeness coincide.

8. Examples

In this section we return to our three main examples of PCAs and associated divergences. We show, in each case, that the strong completeness axiom holds. We shall also discuss some of properties of the examples, particularly in relation to their ability to represent "parallel" and "sequential" computation.

The Kleene PCA

We begin with the PCA $\mathbb{K}$ together with the divergence $\emptyset$. Surprisingly, only the most basic results of recursion theory are needed to show that the three conditions of Proposition 7.4 hold.

(i) We write *ev* for the partial recursive function mapping m to $\{m\}(m)$. Consider the following representation of Σ, which is easily seen to be isomorphic to the original in

Ass($\mathbb{K}$):

$$\|\top\| \;=\; \{e \mid ev(e){\downarrow}\} \qquad\qquad \|\bot\| \;=\; \{e \mid ev(e){\uparrow}\},$$

i.e. $\|\top\|$ is the "halting set" K (see (Cutland 1980, p. 123)) and $\|\bot\|$ is its complement. Suppose there were a morphism $neg : \Sigma \to \Sigma$, tracked by m, that inverted Σ. Then it would hold that $ev(me)\downarrow$ iff $e \notin K$, so the complement of K would be recursively enumerable. But this is not the case.

(ii) Consider the function $f : \mathbb{N} \to |F|$ defined by:

$$f(m)(n) \quad = \quad \begin{cases} \top & \text{if, forall } n' \leq n, \text{ not } T(m, m, n'), \\ \bot & \text{otherwise,} \end{cases}$$

where T is the primitive-recursive predicate, T_1, of Kleene's normal form theorem (Cutland 1980, p. 89). Note that $f(m) = \infty$ if and only if $ev(m){\uparrow}$. It is easy to see that f is recursive. Now suppose, for contradiction, that there were a morphism $inf\colon F \to \Sigma$, tracked by l, mapping ∞ to $\top$ and everything else to $\bot$. Then it would hold that $ev(\{l\}(f(m))){\downarrow}$ iff $m \notin K$. So again the complement of K would be recursively enumerable — a contradiction.

(iii$'$) It is convenient to take the following isomorphic representation of $L\mathbf{2}$.

$$\|\ulcorner tt \urcorner\| \;=\; \{e \mid ev(e) = true\} \qquad \|\ulcorner f\!f \urcorner\| \;=\; \{e \mid ev(e) = false\} \qquad \|\bot\| \;=\; \{e \mid ev(e){\uparrow}\}.$$

Let f be a recursive function mapping each n to a realizer for n in I (such a function is easily found). Let g be a partial recursive function which given an input e computes in parallel (by interleaving computations) $ev(\{e\}(f(0)))$, $ev(\{e\}(f(1)))$, $ev(\{e\}(f(2)))\ldots$ until one of the following occurs: some $ev(\{e\}(f(n)))$ terminates with the value *true*, in which case $g(e) = true$; or some $ev(\{e\}(f(n)))$ terminates with the value *false*, in which case $g(e) = false$; or no $ev(\{e\}(f(n)))$ terminates with such a value, in which case $g(e)$ is undefined. Now if e tracks some function in $|(L\mathbf{2})^I|$ then $ev(\{e\}(f(n))) = true$ implies $ev(\{e\}(f(n+1))) = true$ (by Lemma 6.3) and similarly $ev(\{e\}(f(n))) = false$ implies $ev(\{e\}(f(n+1))) = false$. Therefore g tracks a partial function from $(L\mathbf{2})^I$ to $\mathbf{2}$ whose corresponding function in $(L\mathbf{2})^I \to L\mathbf{2}$ has the required properties. Thus Axiom 2 is indeed satisfied in **Ass**($\mathbb{K}$).

The proof above that (iii$'$) holds demonstrates the technique of interleaving computations in order to execute them in parallel. Such techniques can be used to interpret the various "parallel" operators defined in Plotkin's PCF paper (Plotkin 1977). Indeed the full parallel PCF of Section 5 of (Plotkin 1977) can be interpreted in the type hierarchy over LN in **WC**($\mathbb{K}$). In Chapter 7 of (Longley 1995) it is shown that this interpretation is both universal and fully abstract.

Semi-sensible λ-theories

Next we consider (total) PCAs of closed λ-terms quotiented by semi-sensible λ-theories, with the divergence of unsolvable terms (example (iii) of Section 4). We shall show that the completeness axiom holds for any semi-sensible λ-theory satisfying an additional property defined below. Although we do not know how to deal with an arbitrary semi-sensible λ-theory, our treatment will include the most natural ones.

Recall the definition of the Böhm tree $BT(M)$ of a λ-term M (Barendregt 1984, Ch. 10). We write: $\perp$ for the unsolvable Böhm tree; $\subseteq$ for the inclusion order on Böhm trees (Barendregt 1984, §10.2); and $=_\eta$ for the standard (infinitary) notion of η-equivalence on Böhm trees (Barendregt 1984, p. 240). Let $\mathcal{B}$ be the semi-sensible theory equating all terms with the same Böhm tree (Barendregt 1984, p. 425), and let $\mathcal{K}^*$ be the maximal sensible λ-theory (Barendregt 1984, §16.2). We write $=_\beta$ for the standard notion of β-equality between λ-terms.

Definition 8.1. (K-stability) A semi-sensible λ-theory $\mathcal{T}$ is said to be **K**-*stable* if whenever $\mathcal{T} \vdash M = \lambda xy.\, x$ and $BT(M) = BT(N)$ it holds that $\mathcal{T} \vdash N = \lambda xy.\, x$.

Lemma 8.2. If $\mathcal{T}$ is **K**-stable then whenever $\mathcal{T} \vdash M = \lambda xy.\, y$ and $BT(M) = BT(N)$ it holds that $\mathcal{T} \vdash N = \lambda xy.\, y$.

PROOF Suppose that $\mathcal{T} \vdash M = \lambda xy.\, y$ and $BT(M) = BT(N)$. Then $\mathcal{T} \vdash \lambda xy.\, Myx = \lambda xy.\, x$ and $BT(\lambda xy.\, Myx) = BT(\lambda xy.\, Nyx)$. So, as $\mathcal{T}$ is **K**-stable, $\mathcal{T} \vdash \lambda xy.\, Nyx = \lambda xy.\, x$, whence $\mathcal{T} \vdash \lambda xy.\, Nxy = \lambda xy.\, y$. As $\mathcal{T}$ is semi-sensible, N is solvable and therefore $N =_\beta \lambda x_1 \ldots x_n.\, x_i(N_1) \ldots (N_k)$ where $k \geq 1$ and $1 \leq i \leq n$. Suppose, for contradiction, that $n = 1$. Then $\lambda xy.\, Nxy =_\beta \lambda xy.\, x(N_1) \ldots (N_k)y$, whence $\mathcal{T} \vdash \lambda xy.\, x(N_1) \ldots (N_k)y = \lambda xy.\, y$. But this equation is easily seen to be inconsistent. Therefore $n \geq 2$. It follows that $\lambda xy.\, Nxy =_\beta N$. Thus indeed $\mathcal{T} \vdash N = \lambda xy.\, y$. □

It is not true that every semi-sensible λ-theory is **K**-stable, but the following proposition shows that several natural ones are.

Proposition 8.3. Let $\mathcal{T}$ be a semi-sensible theory, then either of the following conditions is sufficient for $\mathcal{T}$ to be **K**-stable:
(i) $\mathcal{T} \subseteq \mathcal{B}$;
(ii) $\mathcal{T} \supseteq \mathcal{B}$.

PROOF (i) Suppose $\mathcal{T} \subseteq \mathcal{B}$. If $\mathcal{T} \vdash M = \lambda xy.\, x$ and $BT(M) = BT(N)$ then $BT(N) = BT(\lambda xy.\, x)$. So, by the construction of the Böhm-tree for N (Barendregt 1984, Ch. 10), $N =_\beta \lambda xy.\, x$. Therefore $\mathcal{T} \vdash N = \lambda xy.\, x$.

(ii) Suppose $\mathcal{T} \supseteq \mathcal{B}$ and $\mathcal{T} \vdash M = \lambda xy.\, x$. Then $BT(M) = BT(N)$ implies $\mathcal{T} \vdash M = N$. So indeed $\mathcal{T} \vdash N = \lambda xy.\, x$. □

Henceforth, let $\mathcal{T}$ be any **K**-stable semi-sensible λ-theory. We consider the total PCA $\Lambda^0/\mathcal{T}$ and we take D to be the divergence $\{[M] \in \Lambda^0/\mathcal{T} \mid M \text{ unsolvable}\}$. We show that the three conditions of Proposition 7.4 hold. The proof uses several nontrivial results about the untyped λ-calculus from (Barendregt 1984).

(i) We consider the following representation of Σ, which is easily seen to be isomorphic to the original in $\mathbf{Ass}(\Lambda^0/\mathcal{T})$:

$$\|\top\| \;=\; \{[\mathbf{I}]\} \qquad\qquad \|\perp\| \;=\; D,$$

where $\mathbf{I} = \lambda x.\, x$. Suppose, for contradiction, that the inversion $neg : |\Sigma| \to |\Sigma|$ is tracked by L. Then $\mathcal{T} \vdash L(\mathbf{\Omega}) = \mathbf{I}$ and $L(\mathbf{I})$ is unsolvable. Hence $BT(L(\mathbf{\Omega})) \not\subseteq BT(L(\mathbf{I}))$. But $BT(\mathbf{\Omega}) \subseteq BT(\mathbf{I})$, so we have contradicted the monotonicity of application relative to $\subseteq$ on Böhm trees (Barendregt 1984, §14.3).

(ii) For each $n \geq 0$ let $\mathbf{Y_n}$ be the term $\lambda f.\, \overbrace{f(\ldots f(\mathbf{\Omega})\ldots)}^{n}$ and let $\mathbf{Y}$ be the usual Curry fixpoint operator $\lambda f.\,(\lambda x.\,f(xx))(\lambda x.\,f(xx))$. Note that $BT(\mathbf{Y_0}) \subseteq BT(\mathbf{Y_1}) \subseteq$ is an ascending chain of Böhm trees with supremum $BT(\mathbf{Y})$.

Consider $\mathbf{H} = \lambda x.\,x(\lambda p\,m.\,\mathit{if}\,(\mathit{iszero}\ m)\ \mathbf{I}\,(p\,(\mathit{pred}\,m)))$. It is easily checked that, for any n, $[\mathbf{H}(\mathbf{Y_n})] \in \|n\|_F$ and $[\mathbf{H}(\mathbf{Y})] \in \|\infty\|_F$ (note that the argument of x in $\mathbf{H}$ tracks $\mathit{up} : F \to F$). Now suppose that L tracks a morphism $f : F \to \Sigma$ such that, for all n, $f(n) = \bot$. Then, for all n, $BT(L(\mathbf{H}(\mathbf{Y_n}))) = \bot$. So, by the continuity of application relative to $\subseteq$ on Böhm trees, Theorem 14.3.22 of (Barendregt 1984), $BT(L(\mathbf{H}(\mathbf{Y}))) = \bot$. Thus indeed f cannot be the forbidden morphism $\mathit{inf} : F \to \Sigma$.

(iii′) It is convenient to take the following representation of $L\mathbf{2}$ in $\mathbf{Ass}(\Lambda^0/\mathcal{T})$:

$$\|\ulcorner tt \urcorner\| \;=\; \{[\lambda xy.\,x]\} \qquad \|\ulcorner f\!f \urcorner\| \;=\; \{[\lambda xy.\,y]\} \qquad \|\bot\| \;=\; D.$$

We shall show that the required morphism $\bigsqcup : (L\mathbf{2})^I \to L\mathbf{2}$ is tracked by the element $\mathbf{L} = \lambda f.\,f(\mathbf{H}(\mathbf{Y}))$, where $\mathbf{H}$ and $\mathbf{Y}$ are as above.

Suppose that M tracks $p : I \to L\mathbf{2}$. Suppose there exists n such that $p(n) = \ulcorner tt \urcorner$. Then $\mathcal{T} \vdash M(\mathbf{H}(\mathbf{Y_n})) = \lambda xy.\,x$, so $\mathcal{K}^* \vdash M(\mathbf{H}(\mathbf{Y_n})) = \lambda xy.\,x$ (by Lemma 17.1.1 of (Barendregt 1984)), whence $BT(M(\mathbf{H}(\mathbf{Y_n})) =_\eta BT(\lambda xy.\,x)$ (by Theorem 16.2.7 of (Barendregt 1984)). By the monotonicity of application relative to $\subseteq$, we have that, for all $m \geq n$, it holds that $BT(M(\mathbf{H}(\mathbf{Y_n})) \subseteq BT(M(\mathbf{H}(\mathbf{Y_m}))$. But, by the definition of $=_\eta$ (Barendregt 1984, p. 240), it is easily checked that for any Böhm trees $T_1 \subseteq T_2$ such that $T_1 =_\eta BT(\lambda xy.\,x)$ it holds that $T_1 = T_2$. Therefore, for all $m \geq n$, it holds that $BT(M(\mathbf{H}(\mathbf{Y_n})) = BT(M(\mathbf{H}(\mathbf{Y_m}))$. So, by the continuity of application, $BT(M(\mathbf{H}(\mathbf{Y_n})) = BT(M(\mathbf{H}(\mathbf{Y}))$. It then follows from the $\mathbf{K}$-stability of $\mathcal{T}$ that $\mathcal{T} \vdash M(\mathbf{H}(\mathbf{Y})) = \lambda xy.\,x$. Thus indeed $\mathbf{L}(M) \in \|\ulcorner tt \urcorner\|$ as required.

Similarly, if there exists n such that $p(n) = \ulcorner f\!f \urcorner$, then the same argument together with Lemma 8.2 shows that $\mathbf{L}(M) \in \|\ulcorner f\!f \urcorner\|$.

Finally, suppose that $p(n) = \bot$ for all n. Then, for all n, $BT(M(\mathbf{H}(\mathbf{Y_n})) = \bot$. So, by the continuity of application, $BT(M(\mathbf{H}(\mathbf{Y})) = \bot$. Thus indeed $\mathbf{L}(M) \in \|\bot\|$, concluding the proof that Axiom 2 holds.

Note the difference in flavour between the above proof that Axiom 2 holds and the earlier proof for $\mathbf{Ass}(\mathbb{K})$. Indeed there is no way to interleave computations in $\mathbf{Ass}(\Lambda^0/\mathcal{T})$. In fact it seems that $\mathbf{Ass}(\Lambda^0/\mathcal{T})$ provides a good model of "sequential" functional computation. A simple demonstration of this is that there is no analogue of the parallel convergence tester of (Abramsky and Ong 1993):

Proposition 8.4. There is no morphism from $\Sigma \times \Sigma$ to Σ representing disjunction.

Proof If disjunction were tracked by M, it would hold that $\mathcal{T} \vdash M\mathbf{\Omega}\mathbf{I} = M\mathbf{I}\mathbf{\Omega} = \mathbf{I}$ and that $M\mathbf{\Omega}\mathbf{\Omega}$ is unsolvable. This contradicts Berry's sequentiality theorem (Barendregt 1984, §14.4). □

(In the case of $\mathbf{Ass}(\Lambda^0/\mathcal{B})$ this result first appeared in (Phoa 1994).) The above proposition shows that only the "sequential" functions from $\Sigma \times \Sigma$ to Σ exist in $\mathbf{Ass}(\Lambda^0/\mathcal{T})$. In fact a much stronger statement holds, for first-order types every element in $\mathbf{Ass}(\Lambda^0/\mathcal{B})$ is definable in the original "sequential" PCF (Plotkin 1977).

Theorem 8.5. (Longley 1995, Theorem 7.4.3) If $\mathcal{T} \subseteq \mathcal{B}$ then every element of $(LN)^n \to LN$ is definable in PCF.

(A similar result should hold for more general $\mathcal{T}$, but the condition in the theorem serves to simplify the proof.) This theorem gives universality for the first-order fragment of PCF and full abstraction for the second-order fragment. Still it is plausible that much more might hold. It is possible that the natural interpretation of the whole of PCF in $\mathbf{Ass}(\Lambda^0/\mathcal{T})$ is universal, and hence fully abstract. This possibility is formulated as a conjecture in (Longley 1995, §7.4), where it is discussed in detail.

We believe that similar results to the above should hold for other λ-term PCAs. We think it would be of particular interest to investigate more operationally defined PCAs, based on, for example, the call-by-name or call-by-value λ-calculi (Plotkin 1975). It is also plausible that other λ-term PCAs might make it easier to prove universality and full-abstraction results.

Dcpo models

Let A be any nontrivial dcpo such that $[A \to A] \lhd A$ and let D be the divergence $\{\bot\}$. We show that the three conditions of Proposition 7.4 hold.

(i) Take the representation of Σ given by $\|\top\| = \{\mathbf{i}\}$ and $\|\bot\| = \{\bot\}$. Clearly there is no morphism $neg : \Sigma \to \Sigma$ inverting Σ as it could only be tracked by an element of A representing a non-monotonic function.

(ii) To show that $inf : F \to \Sigma$ cannot exist, we shall find an ascending chain of elements $a_0, a_1, \ldots \in A$ such that $a_n \in \|n\|_F$ and $\bigsqcup_i a_i \in \|\infty\|_F$. It then follows that *inf* cannot exist because it could only be tracked by an element of A representing a non-continuous function.

Recall that a subset $B \subseteq A$ is *Scott-closed* if it is down-closed (i.e. if $b \in B$ and $a \leq b$ then $a \in B$) and, for all directed $B' \subseteq B$, it holds that $\bigsqcup B' \in B$. For each n let C_n be the smallest Scott-closed subset of A containing $\{\overline{n}, \overline{n+1}, \ldots\}$. Clearly $C_n \supseteq C_{n+1}$. For any n, it is easy to find an element $a_n \in A$ such that $a_n\, \overline{n} = i$ and, for all $m > n$, $a_n\, \overline{m} = \bot$. Therefore $\overline{n} \notin C_{n+1}$ (as the set $\{b \in A \mid a_n b = \bot\}$ is Scott-closed). For each n define $c_n : A \to A$ by:

$$c_n(a) \quad = \quad \begin{cases} \bot & \text{if } a \in C_n, \\ \mathbf{i} & \text{otherwise.} \end{cases}$$

The c_n are all continuous and $c_0 \sqsubseteq c_1 \ldots \sqsubseteq$ is an ascending chain in $[A \to A]$. Its supremum c_∞ satisfies: $c_\infty(a) = \bot$ if $a \in C_\infty$ where $C_\infty = \bigcap_n C_n$; and $c_\infty(a) = \mathbf{i}$ if $a \notin C_\infty$. The required elements $a_0, a_1, \ldots$ and a_∞ are then obtained as the image of $c_0, c_1, \ldots$ and c_∞ under the embedding of $[A \to A]$ in A.

(iii$'$) Take the following representation of $L\mathbf{2}$ in $\mathbf{Ass}(A)$:

$$\|\ulcorner tt \urcorner\| \;=\; \{true\} \qquad \|\ulcorner f\!f \urcorner\| \;=\; \{false\} \qquad \|\bot\| \;=\; \{\bot\}.$$

It follows easily by monotonicity and continuity considerations that the required morphism $\bigsqcup : (L\mathbf{2})^I \to L\mathbf{2}$ is tracked by $\lambda^* f.\, f\, a_\infty$ where a_∞ is as above. Thus Axiom 2 does indeed hold.

Not surprisingly, in this case $\mathbf{Ass}(A)$ can interpret all the parallel operators of (Plotkin 1977), although this time for "denotational" rather than "algorithmic" reasons. The case where A is Scott's λ-model $\mathcal{P}\omega$ (Barendregt 1984, §18.1) is considered in (Longley 1995), where a full abstraction result for parallel PCF is obtained. Universality and full abstraction results are also obtained for the effective subPCA $\mathcal{P}\omega_{re}$.

9. The intrinsic order

So far the development has been entirely order-free (apart from the occasional remark on monotonicity). In this section we show how everything we have done is compatible with an order-theoretic approach. We shall identify a subcategory of **Ass** of intrinsically ordered objects, and show that its restriction to **WC** satisfies all the closure properties of **WC**. There are three reasons for doing this. First, the closure of the subcategory under all the operations means that an intrinsic partial order is present on all objects defined using the operations. The presence of this partial order can be useful for reasoning about properties of the objects. Second, there are possible conceptual reasons for wishing to restrict to such a category. Third, we shall offer an alternative, illuminating characterization of the complete intrinsically-ordered objects.

The intrinsic order will be determined by morphisms into Σ. Indeed such morphisms determine a preorder on any assembly.

Definition 9.1. (Σ-preorder) The *Σ-preorder on X* is the relation $\preceq_X$ on $|X|$ defined by

$$x \preceq_X x' \quad \text{iff} \quad \text{for all } p : X \to \Sigma,\, p(x) = \top \text{ implies } p(x') = \top.$$

It is clear that $\preceq$ is indeed a preorder. Further, it is straightforward to show that $\preceq$ is preserved by any morphism in **Ass**, i.e. for any $f : X \to Y$ it holds that $x \preceq_X x'$ implies $f(x) \preceq_Y f(x')$.

Intuitively, one thinks of a map $p : X \to \Sigma$ as a test on X. An element $x \in X$ passes the test if $p(x) = \top$. As $\top \in \Sigma$ corresponds to termination, one can observe if x passes such a test; but failure cannot be similarly observed because $\bot$ corresponds to nontermination. Thus the notion of Σ-test is akin to the notion of semidecidable property, and indeed in the case of $\mathbb{K}$ the Σ-tests on N are exactly the semidecidable properties of N. It is natural to consider the subcategory of those objects whose elements are determined by the set of tests they satisfy. Such objects are conceptually appealing because any two distinct elements can be distinguished by some observational test that one satisfies but the other does not.

Definition 9.2. (Σ-poset) We say that an assembly X is a *Σ-poset* if $\preceq_X$ is a partial order.

The notion of Σ-poset first appears in (Phoa 1990), where the term *Σ-space* is used for the Σ-posets in $\mathbf{Mod}(\mathbb{K})$.[‡] We write $\mathbf{Mod}_{\boldsymbol{\Sigma}}$ for the full subcategory of **Ass** consisting of those assemblies that are Σ-posets. This is justified by:

Theorem 9.3. The category $\mathbf{Mod}_{\boldsymbol{\Sigma}}$ is a full subcategory of **Mod**. It is bicartesian-closed with finite limits and nno and the inclusion of $\mathbf{Mod}_{\boldsymbol{\Sigma}}$ in **Ass** preserves this structure. Further, $\mathbf{Mod}_{\boldsymbol{\Sigma}}$ is an exponential ideal of **Ass** which is closed under lifting and under subobjects.

Proof First, suppose we have a Σ-poset X with $x, y \in |X|$ such that $a \in \|x\| \cap \|y\|$. Then, for any $p : X \to \Sigma$, it holds that $p(x) = p(y)$. So $x \preceq y$ and $y \preceq x$. Therefore $x = y$, because X is a Σ-poset. Thus indeed X is modest.

Next we show that $\mathbf{Mod}_{\boldsymbol{\Sigma}}$ is closed under subobjects. Suppose we have a mono $m : X \to Y$ where Y is a Σ-poset.To show that X is a Σ-poset, suppose $x \preceq x'$ and $x' \preceq x$. As $\preceq$ is preserved by arbitrary morphisms we have that $m(x) \preceq m(x')$ and $m(x') \preceq m(x)$. Therefore $m(x) = m(x')$ because Y is a Σ-poset. Thus indeed $x = x'$ because m is mono.

As $\mathbf{Mod}_{\boldsymbol{\Sigma}}$ is closed under subobjects it is closed under equalizers. It also trivially contains the terminal object. For binary products we show that:

$$(x, y) \preceq_{X \times Y} (x', y') \quad \text{if and only if} \quad x \preceq_X x' \text{ and } y \preceq_Y y'.$$

Thus $X \times Y$ is clearly a Σ-poset whenever both X and Y are (in fact the left-to-right implication alone is sufficient for this). The left-to-right implication is just the preservation of $\preceq$ by the projections π_1 and π_2. For the converse suppose that $x \preceq x'$ and $y \preceq y'$. Suppose that $p : X \times Y \to \Sigma$ is such that $p(x, y) = \top$. Then $p(x', y) = \top$, because $x \preceq x'$, whence $p(x', y') = \top$, because $y \preceq y'$. Thus indeed $(x, y) \preceq (x', y')$.

For closure under finite coproducts, one shows very easily that the partial order on $X + Y$ is the poset coproduct of the partial order on X with that on Y. For the nno, it is straightforward to show that $\preceq$ is the discrete partial order.

For the exponential ideal, suppose that X is a Σ-poset and Y is arbitrary. It is easy to show that:

$$f \preceq_{X^Y} f' \quad \text{implies} \quad \text{forall } y \in |y|,\ f(y) \preceq_X f'(y). \tag{1}$$

To show that X^Y is a Σ-poset, suppose that $f \preceq f'$ and $f' \preceq f$. Then, by (1), we have that, forall $y \in |y|$, $f(y) \preceq f'(y)$ and $f'(y) \preceq f(y)$. So $f(y) = f'(y)$, because X is a Σ-poset. Thus indeed $f = f'$.

Finally, for closure under lifting, we first show that:

$$e \preceq_{LX} e' \quad \text{implies} \quad \begin{array}{l}\text{either } e = \bot \text{ or there exist } x, x' \in |X| \text{ such that } e = \ulcorner x \urcorner, \\ e' = \ulcorner x' \urcorner \text{ and } x \preceq_X x'.\end{array} \tag{2}$$

Suppose that $e \preceq e'$ and $e \neq \bot$. Then $e = \ulcorner x \urcorner$ for some x. Now $\chi(e) = \top$, so $\chi(e') = \top$, because $e \preceq e'$. Therefore $e' = \ulcorner x' \urcorner$ for some x'. To see that $x \preceq x'$ suppose $p : X \to \Sigma$ is such that $p(x) = \top$. Then $\mu \circ Lp(\ulcorner x \urcorner) = \top$ so $\mu \circ Lp(\ulcorner x' \urcorner) = \top$, again because $e \preceq e'$. Thus $p(x') = \top$, establishing that indeed (2) holds.

[‡] Warning! In (Reus 1995) the term Σ-poset is used to mean *extensional* in the terminology of Section 5. We prefer our usage because a Σ-poset is precisely what it says it is.

Now suppose that X is a Σ-poset. To show that LX is, suppose that $e \preceq_{LX} e'$ and $e' \preceq_{LX} e$. Then, by (2), either $e = e' = \bot$, or $e = \ulcorner x \urcorner$ and $e' = \ulcorner x' \urcorner$ where $x \preceq_X x'$ and $x' \preceq_X x$. So either way $e = e'$ as required. □

The proof of the theorem shows that the orders on $X \times Y$, $X + Y$ and N are the expected ones inherited from the category of posets. On the other hand we have not shown that the order on X^Y is the expected pointwise order, i.e. that the converse implication to (1) holds. We shall return to this point below. Also we have not yet shown that the order on LX is the expected lifted order, i.e. that the converse to (2) holds. This requires a familiar condition on Σ.

Proposition 9.4. If there is no morphism $neg : \Sigma \to \Sigma$ such that $neg(\top) = \bot$ and $neg(\bot) = \top$ then it holds that $e \preceq_{LX} e'$ if and only if either $e = \bot$ or there exist $x, x' \in |X|$ such that $e = \ulcorner x \urcorner$, $e' = \ulcorner x' \urcorner$ and $x \preceq_X x'$.

PROOF The left-to-right implication holds anyway, as in the proof of Theorem 9.3. For the converse we show firstly that, for any $e \in |LX|$, it holds that $\bot \preceq_{LX} e$. Suppose then that $p : LX \to \Sigma$ is such that $p(\bot) = \top$. Let $l : \Sigma \to LX$ be the evident map such that $l(\bot) = \bot$ and $l(\top) = e$. Then necessarily $p(e) = \top$, because otherwise $p \circ l$ would be the forbidden inversion on Σ. Thus indeed $\bot \preceq_{LX} e$. It remains to show that $x \preceq_X x'$ implies $\ulcorner x \urcorner \preceq_{LX} \ulcorner x' \urcorner$. But this is just the preservation of $\preceq$ by $\eta : X \to LX$. □

It is easy to see that the non-existence of an inversion map on Σ is also a necessary condition for the above characterization of $\preceq_{LX}$ to hold. Recall from Proposition 6.1 that this condition is satisfied whenever Axiom 1 holds.

Another consequence of Theorem 9.3 is that Σ, I, F and $L2$ are all Σ-posets. It follows from Proposition 9.4 that, when Σ has no inversion, the Σ-order on these objects coincides with the natural order defined in Section 6. Thus, by Lemma 6.3, the Σ-order on these objects also coincides with the path relation. We mention however that this coincidence does not mean that these objects are necessarily *linked* in the sense of (Phoa 1990) (an object is linked if such a coincidence holds in a suitable internal sense). For example, Σ is not linked in any of the $\mathbf{Ass}(\Lambda^0/\mathcal{T})$ examples of Section 8, essentially because $L(\Sigma) \not\cong \Sigma^\Sigma$ (which is a consequence of Proposition 8.4). The problem with the failure of linkedness is that the only known way of proving that the Σ-order on exponentials is pointwise depends on objects being linked, see (Phoa 1990). We believe that it should be possible to show that exponentials are ordered pointwise for a good class of objects in $\mathbf{Ass}(\Lambda^0/\mathcal{T})$. However, how to do this remains an intriguing open question.[§]

We now move on to discussing how the intrinsic order combines with the notion of predomain. We write $\mathbf{WC}_\Sigma$ for the full subcategory of **Ass** consisting of well-complete Σ-posets. Clearly Theorems 7.2, 7.5 and 9.3 all combine to give:

Theorem 9.5. If Axiom 2 holds then $\mathbf{WC}_\Sigma$ is bicartesian-closed with finite limits and nno and the inclusion in **Ass** preserves this structure. Further, $\mathbf{WC}_\Sigma$ is an exponential ideal of **Ass** which is closed under lifting.

[§] If a universality result holds for PCF, as discussed beneath Theorem 8.5, then it will follow from Milner's context lemma (Milner 1977) that the Σ-order is pointwise on those objects that interpret PCF types.

Thus one can perfectly well take $\mathbf{WC}_\Sigma$ as a category of predomains instead, and nothing important is lost.

We conclude this section with some observations on the notion of completeness for Σ-posets. We have seen that when Axiom 1 holds the Σ-order on I is the natural order. It follows that every I-chain in a Σ-poset X is an ascending chain in the order-theoretic sense. The next theorem shows that, for Σ-posets, the notion of completeness corresponds to a natural form of "effective" chain-completeness in the order-theoretic sense. We shall also see that one of the conditions of Proposition 5.5 can be dropped for Σ-posets.

Theorem 9.6. If Axiom 1 holds then the following are equivalent for a Σ-poset X:
(i) X is complete;
(ii) X has an eventual-value operator;
(iii) Every chain $c : I \to X$ has a least upper bound and there is a morphism $\bigsqcup : X^I \to X$ such that, for all $c \in X^I$, it holds that $\bigsqcup c$ is the lub of $\{c(n) \mid n \in |I|\}$.

PROOF To show that (i) is equivalent to (ii), suppose that $c_1, c_2 : F \to X$ restrict to the same I-chain. By Proposition 5.5 it suffices to show that $c_1 = c_2$. Because X is a Σ-poset, we need only show that, forall $p : X \to \Sigma$, $p \circ c_1 = p \circ c_2$. But given such a p we have that $p \circ c_1$ and $p \circ c_2$ both restrict to the same I-chain. Thus indeed $p \circ c_1 = p \circ c_2$ because Σ is complete.

Clearly (iii) implies (ii) because the lub-finding operation, $\bigsqcup$, is an eventual-value operator.

Lastly, we show that (i) implies (iii). Given any $c : I \to X$, it suffices to show that $\overline{c}(\infty)$ (where $\overline{c} : F \to X$ is the unique morphism extending c) is the lub of $\{c(n) \mid n \in |I|\}$, for then $\bigsqcup_X$ (as in Theorem 5.6(ii)) is the required lub-finding operation. Firstly $\overline{c}(\infty)$ is obviously an upper bound by the monotonicity of $\overline{c}$. To see it is the least such, suppose that $x \in |X|$ is another upper bound. Suppose that $p : X \to \Sigma$ is such that $p(\overline{c}(\infty)) = \top$. By Proposition 6.1(ii), there exists n such that $p(\overline{c}(n)) = \top$. Therefore $p(c(n)) = \top$ so, as $c(n) \preceq x$, $p(x) = \top$. This shows that indeed $\overline{c}(\infty) \preceq x$. □

10. Comparison with other approaches

In this section we consider the other approaches to defining full subcategories of predomains that have been previously adopted in various realizability models. All approaches amount to identifying predomains as objects satisfying some sort of completeness assumption.

One natural approach to defining completeness is to "internalize" the standard definition of ω-cpo, i.e. a poset in which all N-indexed ascending chains have a least upper bound. This approach was originally adopted in (Phoa 1990) (see also (Mitchell and Viswanathan 1995)). For such a definition to make sense it is necessary to restrict to a subcategory of posets, and it is natural to take the Σ-posets for this (called *Σ-spaces* by Phoa). Given a Σ-poset X, an *N-chain* in X is a morphism $c : N \to X$ such that, forall n, $c(n) \preceq c(n+1)$. We say that a Σ-poset X is *N-complete* if: every N-chain c in X has a least upper bound under $\preceq$; and there is an element $r \in A$ such that, whenever p tracks an N-chain c in X, it holds that $rp \in \| \bigsqcup c \|_X$. Phoa used the terminology *complete Σ-space* for N-complete Σ-poset. It follows easily from Theorem 9.6 that if X is an

N-complete Σ-poset then it is a complete Σ-poset. In general the reverse inclusion fails. Intuitively, this is because one can, even in $\mathbf{Ass}(\mathbb{K})$, find Σ-posets with N-chains that do not correspond to I-chains. A complete Σ-poset need not have lubs for such chains whereas an N-complete Σ-poset must. An example of such behaviour appears in the proof of Proposition 10.1 below. The counterexamples in $\mathbf{Ass}(\mathbb{K})$ are of a rather different nature, and are omitted for lack of space.

Phoa showed that in $\mathbf{Ass}(\mathbb{K})$ the N-complete Σ-posets form a full-reflective subcategory closed under all desirable operations on a category of predomains. However, N-complete Σ-posets do not have such good properties in general. As in Section 8, let $\mathcal{B}$ be the theory of Böhm-tree equality on Λ^0 and let D be the divergence of unsolvable terms. A striking demonstration of the inadequacy of N-completeness is given by:

Proposition 10.1. The object Σ is not an N-complete Σ-poset in $\mathbf{Ass}(\Lambda^0/\mathcal{B})$.

PROOF (Sketch) We show first that the object $\Sigma^{LN\times\Sigma}$ is not an N-complete Σ-poset. Let $h : \mathbb{N} \to \mathbb{N}$ be any recursive function whose range is a recursively enumerable non-recursive set. Consider the following family of functions $f_i : |LN \times \Sigma| \to |\Sigma|$ where $0 \le i \le \infty$:

$$f_i(e,p) \quad = \quad \begin{cases} \top & \text{if there exists } n \text{ such that } e = \ulcorner n \urcorner \text{ and either:} \\ & \quad n = h(m) \text{ for some } m < i\text{; or } p = \top, \\ \bot & \text{otherwise,} \end{cases}$$

It is straightforward to show that there is a morphism $c : N \to \Sigma^{LN\times\Sigma}$ displaying the f_n as an N-chain in $\Sigma^{LN\times\Sigma}$. One can show that the supremum, if it existed, of the f_n would have to be f_∞. However, f_∞ cannot be tracked because the natural embedding $\Sigma^{LN\times\Sigma} \lhd LN^{LN\times LN}$ would map f_∞ to an element that is not PCF definable, contradicting Theorem 8.5. (Essentially f_∞ is an example of a "computable sequential" function that that is not "sequentially computable", cf. a famous example of (Trakhtenbrot 1975).)

Now suppose that Σ were N-complete. Let $\hat{c} : LN \times \Sigma \to \Sigma^N$ be obtained from the morphism c above via a twofold exponential transpose. It is easy to see that the range of c is always an N-chain in Σ. By composing the element of $\Lambda^0/\mathcal{B}$ that finds lubs of N-chains in Σ with the element tracking $\hat{c}$, one obtains f_∞ as a morphism $LN \times \Sigma \to \Sigma$, a contradiction. □

It is interesting to observe that even though Σ is N-complete, every N-chain in Σ does still have a lub. The failure of N-completeness is because the lub-finding operation is not tracked. In contrast, as we showed, $\Sigma^{LN\times\Sigma}$ is not N-complete in the stronger sense that there exist N-chains with no lub. Note also that the above proposition shows that the N-complete Σ-posets are not closed under lifting (as $\mathbf{1}$ obviously is an N-complete Σ-poset).

Another category of predomains in $\mathbf{Ass}(\mathbb{K})$ was considered in (Freyd *et al.* 1990). Their category was based on a strengthened notion of Σ-poset. It is easy to show that X is a Σ-poset iff the canonical map $X \to \Sigma^{\Sigma^X}$ is mono. We say that X is *extensional* if this map is in fact a regular mono in $\mathbf{Ass}$. Thus any extensional object is a Σ-poset, but in general not vice-versa. Freyd *et al* considered the category of N-complete extensional objects in $\mathbf{Ass}(\mathbb{K})$ and established for it similar properties to those established for the category of N-complete Σ-posets by Phoa. This category has also been investigated thoroughly in (Reus 1995). Clearly Proposition 10.1 already shows that the category of N-complete extensional objects will not do as a category of predomains in general. However, in $\mathbf{Ass}(\mathbb{K})$ it turns out that the N-complete extensional objects coincide with the well-complete extensional objects. This is because for extensional objects in $\mathbf{Ass}(\mathbb{K})$ the N-chains and I-chains turn out to coincide, see Lemma 2.7.2 of (Reus 1995). Thus it is natural to consider whether the well-complete extensional objects provide a good notion of predomain in general (just as the well-complete Σ-posets do by Theorem 9.5). We do not know the answer to this question, but we strongly believe that:

Conjecture 10.2. The object $\mathbf{2}$ is not extensional in $\mathbf{Ass}(\Lambda^0/\mathcal{B})$.

To attempt a proof of the conjecture one factors the map $\mathbf{2} \to \Sigma^{\Sigma^{\mathbf{2}}}$ as an epi followed by a regular mono. It suffices to prove that the epi, which is easy to construct explicitly, is not an isomorphism. Although we are convinced that there exists no λ-term with the right properties to track an inverse to the epi, we have been unable to prove its non-existence. Incidentally, if the conjecture holds then it follows that N is also not extensional, because extensional objects are closed under retracts.

Perhaps the most interesting alternative category of predomains is the category of *replete* objects due to (Hyland 1990) and (Taylor 1991), see also (Hyland and Moggi 1995). We say a map $f : X \to Y$ is Σ-*equable* if Σ is orthogonal to f. Thus, for example, our Axiom 1 stated that $\iota : I \to F$ was Σ-equable. An object X is said to be *replete* if it is orthogonal to every Σ-equable map. Two remarkable facts about the replete objects are: they form the smallest full-reflective exponential ideal of $\mathbf{Ass}$ that contains Σ; and they are closed under lifting. If Axiom 1 holds then it is clear that every replete object is well-complete. It is also fairly straightforward to show that every replete object is extensional. (For this, take the epi/regular-mono factorization of $X \to \Sigma^{\Sigma^X}$ for a replete X. One easily shows that the epi component is Σ-equable. Then the orthogonality of X to the epi implies that the epi is in fact an isomorphism.) Thus, if Conjecture 10.2 holds, the object $\mathbf{2}$ is not replete in $\mathbf{Ass}(\Lambda^0/\mathcal{B})$. Note that the proof that $\mathbf{2}$ is replete in (Hyland 1990) relies on the presence of parallel features that do not exist in the λ-term models.

From the above discussion, it appears that subcategories of extensional objects need not be closed under finite coproducts in $\mathbf{Ass}$. But this is not necessarily a disaster. Any full reflective subcategory of $\mathbf{Ass}$ (such as the category of replete objects) does have finite coproducts, although they need not be inherited from $\mathbf{Ass}$. One could argue that, for the purposes of domain theory, the mere presence of coproducts, whatever they are, is sufficient. However, it seems to us that the constructions in $\mathbf{Ass}$ provide natural computational representations of datatypes, and one would like to use such representations whenever possible. Thus a significant advantage of well-completeness over repleteness is that it supports the use of such representations in examples like the various models based on λ-term PCAs.

It is also interesting to compare the different philosophies underlying the definitions of repleteness and well-completeness. The category of replete objects can be thought of as the smallest possible reasonable subcategory of predomains (assuming such a category should be a full-reflective exponential ideal containing Σ). However, a consequence of this parsimony is that in some models perfectly good objects are left out. In contrast, the notion of well-completeness represents an attempt to find the *largest* well-behaved full subcategory of predomains. Indeed, this can be mathematically justified to some extent. In the next section we shall sketch why **WC** is a full reflective subcategory of **Ass**. Marcelo Fiore observed that **WC** is therefore the largest full reflective subcategory of **Ass** that is both closed under lifting and for which the morphism $\iota : I \to F$ is reflected to an isomorphism.

We finish this section with a brief mention of some other relevant literature. In (Amadio 1991; Amadio 1992; Abadi and Plotkin 1990) there are various model-specific approaches to extracting a category of predomains from models based on dcpo PCAs. Phoa wrote two papers on predomains in realizability models generated by PCAs other than $\mathbb{K}$. In (Phoa 1992a) he investigated the replete objects in $\mathbf{Ass}(\mathcal{P}\omega_{re})$, based on the effective subPCA of Scott's $\mathcal{P}\omega$ (Barendregt 1984, §18.1). In (Phoa 1994) he gave a seemingly model-specific definition of predomain in $\mathbf{Ass}(\Lambda^0/\mathcal{B})$, which turns out to be equivalent to the category of well-complete objects. It appears that Mitchell and Viswanathan also use the well-complete objects in their category of PERs generated by a call-by-value λ-calculus (Mitchell and Viswanathan 1995).

11. Further developments

In this paper we have proposed a definition of a category of predomains, the *well-complete objects*, that works uniformly across a range of realizability models based on a PCA equipped with a divergence. We have shown that this category is closed under basic constructions. In this section we sketch how the well-complete objects also support more elaborate constructions such as those required for interpreting polymorphism and recursive domain equations.

In order to tackle these more complex constructions it seems convenient use a "synthetic" style of development using the internal logic of the associated realizability toposes. For this section alone, we require that the reader have some familiarity with: toposes and their internal logic (Mac Lane and Moerdijk 1992); realizability toposes (Hyland 1982; Hyland *et al.* 1980); and with small internally complete categories (Hyland 1988; Hyland *et al.* 1990).

Suppose given a PCA, A, and a divergence, D, such that Axiom 2 holds. We sketch why **WC** is equivalent to a small (suitably-)complete internal category of **RT**.¶

As stated in Section 4, Σ is a dominance. In the standard way, see (Rosolini 1986), the morphisms $X \to \Sigma$ determine the Σ-*subsets* of X. Consider the Σ-subset I_p of I characterized by $p : I \to \Sigma$. Because Axiom 1 holds, an associated Σ-subset F_p of F is determined by $\overline{p} : F \to \Sigma$. Moreover, there is an inclusion $\iota_p : I_p \to F_p$. All these constructions can be coded up in the internal logic of the topos. It turns out that an

¶ The parenthetical qualification is due to certain subtleties concerning the notion of completeness, which are discussed thoroughly in (Hyland *et al.* 1990).

object X is well-complete if and only if we have:

$$\mathbf{RT} \models \forall p \in \Sigma^I.\ X \text{ is orthogonal to } \iota_p, \tag{3}$$

i.e. the formula expressing this property holds internally. Therefore well-completeness corresponds to orthogonality to an internally defined family of morphisms. It follows, see (Hyland *et al.* 1990), that the well-complete objects are closed under arbitrary internal limits.

Next we find an internal category equivalent (enough) to **WC**. Consider the small full subcategory, **ModP**, of **Mod** defined in Section 3. As shown in (Hyland 1988; Hyland *et al.* 1990) it is also an internal category of **RT** (indeed of **Ass**) which is internally complete for internal diagrams in **Ass**. Consider the full internal subcategory, **WCP**, of well-complete objects of **ModP** (this is definable because the property of well-completeness is expressible in the internal logic). By the characterization (3) of well-completeness above, **WCP** is as internally complete as **ModP**, hence it is complete for internal diagrams in **Ass**. Also the object of objects of **WCP** is, by its definition, in **Ass**. Therefore **WCP** is complete enough to have internal products indexed over arbitrary families of objects. Hence it is complete enough to interpret polymorphism.

It remains to sketch the interpretation of recursive types. The small completeness of **WCP** already gives an interpretation to covariant inductive and coinductive types, see (Hyland 1988). However, in general one wants solutions to possibly mixed-variance recursive domain equations. Freyd has identified the appropriate structure to look for, i.e. an internally *algebraically compact category* (Freyd 1990; Freyd 1992). The category **WCP** itself cannot be algebraically compact because it is cartesian closed. However, we consider instead the associated category of pointed objects and strict maps, $\mathbf{WCP}_\perp$. Because $\mathbf{WCP}_\perp$ is the (internal) Eilenberg-Moore category of the lift monad on **WCP**, it is as internally complete as **WCP** (by the internal version of Exercise 2 on p. 138 of (Mac Lane 1971)). It follows that every internal endofunctor on $\mathbf{WCP}_\perp$ has an initial algebra and a final coalgebra, see (Hyland 1988). Further, it follows from the theory in (Simpson 1992), that the structure map of each is the inverse of the other. Therefore $\mathbf{WCP}_\perp$ indeed algebraically compact.

Finally, we mention that it follows from small completeness that **WCP** (and hence **WC**) is a full reflective subcategory of **Ass**. The reflection functor is obtained using the internal version of Freyd's Adjoint Functor Theorem (Mac Lane 1971). The solution set condition is trivial because **WCP** is small.

The above discussion is rather sketchy, and it would certainly require another paper to flesh out the details properly. Indeed a paper is under preparation in which the internal logic is used to show that the (computationally more natural) Kleisli category of the lift monad is also (internally) algebraically compact. The treatment is wholly axiomatic and is expected to apply to a range of non-realizability models as well as realizability models based on other forms of realizability (such as modified realizability).

The reader who is unhappy with the use of the internal logic might like to see such a development carried out at the same concrete level as that at which the rest of this paper has been presented. However, it seems that the required calculations with realizers would become rather complicated. Also the focus of the present paper on realizability models lacks the generality of an axiomatic approach. A more general way of avoiding the internal logic would be to adopt an external categorical approach using fibred category theory as in (Hyland *et al.* 1990; Hyland and Moggi 1995).

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